RADICAL EXTENSIONS AND CROSSED CHARACTERS

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E. Witt[1] has given a theory of abelian extensions of fields containing sufficiently many roots of unity which consists essentially, as has been remarked before,[2] in applying the theory of characters of finite abelian groups. There exists now a sufficiently developed theory of crossed characters,[3] and it is the object of this note to show that a fairly complete and simple theory of radical extensions may be obtained if one follows Witt's treatment[4] of abelian extensions, only substituting for the classical theory of characters the theory of crossed characters.[5]

Suppose that the commutative field[6] K is a finite, normal, and separable extension of the field F, that the characteristic of the field K is either 0 or prime to the given integer m, and that E is the group of the mth roots of unity contained[7] in K. The Galois group G of the extension K of F consists of all the F-automorphisms of the field K (automorphisms of the field K which leave the elements in F invariant). Every automorphism g in G induces in E an automorphism which we also denote by g, and the correspondence mapping the automorphism g in G upon the automorphism g of E shall be denoted by C.

A C-character of the group G is a single-valued G to E function $f(g)$, satisfying the functional equation $f(u)^v f(v) = f(uv)$.

LEMMA 1. *The function v^{1-g} of the element g in G, for v an element in K, is a C-character of G if, and only if, v^m is an element, not 0, in F.*

PROOF. If v^{1-g} is a C-character of G, then v^{1-g} is for every g in G an element in E, so that $(v^m)^{1-g} = 1$ for every g in G. Thus v^m is a fixed element of the Galois group of K over F, proving that v^m is an element, not 0, in F.

Assume conversely that $v^m \neq 0$ is in F. Then $(v^{1-g})^m = (v^m)^{1-g} = 1$ for

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[1] Witt [1], [2]; the references refer to the bibliography at the end of the paper.

[2] Baer [1].

[3] Baer [2].

[4] Or the treatment as suggested in Baer [1].

[5] The importance of the theory of crossed characters for the theory of radical extensions has recently been stressed by MacLane-Schilling [1].

[6] As all the fields will be commutative, we shall omit the word "commutative" in the future.

[7] K need not contain m distinct mth roots of unity; cp. Theorem 3 below.

every g in G, so that $v^{1-\sigma}$ is a single-valued G to E function. If s, t are elements in G, then $v^{1-\sigma t}=v^{1-t}v^{t-\sigma t}=(v^{1-\sigma})^t v^{1-t}$, and we have shown that $v^{1-\sigma}$ is a C-character of G.

We denote, as customary, by F^* the multiplicative group of the elements, not 0, in F, and it will be convenient to denote by K_m the multiplicative group of those elements in K whose mth power is in F^*. Clearly F^* is a subgroup of K_m.

THEOREM 1. *An isomorphism of the group K_m/F^* upon the group of all the C-characters of G is effected by mapping the element v in K_m upon the function $v^{1-\sigma}$ of the elements in G.*

PROOF. It is a consequence of Lemma 1 that $v^{1-\sigma}$ is a C-character of G whenever v is an element in K_m, and it is readily seen that a homomorphism of K_m into the group of all the C-characters of G is effected by mapping v upon the function $v^{1-\sigma}$ of the elements in G. Furthermore $v^{1-\sigma}=1$ for every g in G if, and only if, v is an element in F^*, as follows from the fundamental theorem of Galois theory, and this shows that exactly the elements in F^* are mapped upon the C-character 1. To complete the proof of our theorem we need show only that[8] every C-character of G may be represented in the form $v^{1-\sigma}$ for some v in K_m. Suppose therefore that $f(g)$ is a C-character of G. Assume that there did not exist an element w in K such that $\sum_{\sigma \text{ in } G} w^\sigma f(g)\neq 0$. Then $\sum_{\sigma \text{ in } G} (w^h)^\sigma f(g)=0$ for every h in G and every w in K. Since the elements $f(g)$ are different from 0 as elements in E, this implies that the determinant $D(w)=\left|w^{h\sigma}\right|=0$ for every w in K. But this is impossible, since the normal basis theorem[9] assures the existence of elements w in K such that $D(w)\neq 0$. Consequently there exists an element w in K such that $v=\sum_{\sigma \text{ in } G} w^\sigma f(g)\neq 0$. If h is some element in G, then

$$v^h f(h)=\sum_{\sigma \text{ in } G} w^{\sigma h} f(g)^h f(h)=\sum_{\sigma \text{ in } G} w^{\sigma h} f(gh)=v,$$

since f is a C-character and gh ranges over G with g. Since $v\neq 0$, we find that $v^{1-h}=f(h)$ for every h in G, and it is a consequence of Lemma 1 that v belongs to K_m, as was to be shown.

The following restatement of Theorem 1 will prove helpful: If V is a coset of K_m/F^*, then $V^{1-\sigma}$ is a C-character of G, and every C-character of G may be represented in one and only one way in the form $V^{1-\sigma}$.

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[8] The following arguments are essentially a restatement of a proof of Speiser [1] p. 3.

[9] Cp. Deuring [1].

We recall that a group B of C-characters of G is termed[10] *complete* if $B(g)=1$ implies $g=1$.

THEOREM 2. *If M is a multiplicative group between F^* and K_m, then the following two properties of M are equivalent:*

(i) $K=F(M)$.

(ii) *The group of the C-characters $V^{1-\sigma}$ for V in M/F^* is complete.*

PROOF. The statement (ii) is obviously equivalent to the fact that 1 is the only $F(M)$-automorphism of K, and that this fact is equivalent to (i) is readily deduced from the Galois theory.

The group G has been termed[11] *C-complete* if 1 is the only element in G which is mapped upon 1 by every C-character of G. The following statement is an immediate consequence of Theorem 2.

COROLLARY 1. *The group G is C-complete if, and only if, K may be obtained by adjoining to F mth roots of elements in F.*

The C-characters of G which have the form $e^{1-\sigma}$ for suitable e *in the group* E have been termed *principal* C-characters of G, and the group of principal C-characters is called the *principal genus*. A complete group of C-characters of G is said to be *strictly complete* if it contains the principal genus.[12]

COROLLARY 2. *If M is a multiplicative group between F^* and K_m, then the following two conditions are necessary and sufficient for the group of C-characters of the form $V^{1-\sigma}$ with V in M/F^* to be strictly complete:*

(a) $K=F(M)$;

(b) *M is the set of all the elements in K whose mth powers are in the subgroup M^m of F^*.*

PROOF. It is readily seen that condition (b) is equivalent to the inequality $E\leqq M$, and hence (b) is true if, and only if, the group of C-characters of the form $V^{1-\sigma}$ for V in M/F^* contains the principal genus. Thus Corollary 2 is an immediate consequence of Theorem 2.

The importance of Corollary 2 stems from the comparative rarity of strictly complete groups of C-characters which are different from the group of all the C-characters.[13]

If M is a group between F^* and K_m, then M^m is a group between F^{*m} and $K_m^m\subseteq F^*$. However, the groups M/F^* and M^m/F^{*m} need not be isomorphic, since the homomorphism $x\to x^m$ maps upon 1 all the mth roots of unity in M, and these need not be in F^*.

[10] Baer [2] introduction to chapter IV.

[11] Baer [2] I.3.

[12] Baer [2] introduction to chapter IV.

[13] Baer [2] Corollary IV.4.2.

In an investigation of extensions by mth roots it is clearly no loss of generality to assume that m has been chosen as small as possible. Then the positive integer m has been determined in such a way that $K=F(K_m)$, though $F(K_{m'})<K$ for every proper divisor m' of m, and this signifies in group-theoretical language that m is the maximum order of the elements in the multiplicative abelian group K_m/F^*. If we use the notation $m=\prod_p p^{m(p)}$, where the product ranges over all the prime numbers p and where almost all the exponents $m(p)$vanish, then the maximum order of the elements in K_m/F^* is m if, and only if, K_m/F^* contains elements of order $p^{m(p)}$ for every p, since the orders of the elements in K_m/F^* are clearly divisors of m.

THEOREM 3. *The following conditions are necessary and sufficient for m to be the maximum order of the elements in K_m/F^*:*

(a) *K contains m distinct mth roots of unity, so that m is the order of E.*

(b) *If $0<m(p)$, and if a pth root of unity different from 1 is contained in F, then K_m^m is not part of F^{*p}*

PROOF. If condition (a) were not satisfied by K, then the order m' of E would be a proper divisor of m; and if w is an element in K_m, g an F-automorphism of K, then $(w^{1-g})^{m'}=1$, since w^{1-g} belongs to E, proving that $w^{m'}=(w^{m'})^g$ for every F-automorphism g of K, and that therefore every $w^{m'}$, for w in K_m, is in F^*. Hence the maximum order of the elements in K_m/F^* is a divisor of the proper divisor m' of m, proving the necessity of (a).

To prove the necessity of (b) we assume that (b) does not hold for some particular prime p. Then $0<m(p)$, F contains p distinct pth roots of unity (so that every pth root of unity is in F) and $K_m^m\subseteq F^{*p}$; and we are going to show that K_m/F^* does not contain elements of order $p^{m(p)}$. Suppose namely that b is an element in K such that $b^{p^{m(p)}}$ is in F^*. Then there exists an integer r, prime to p, such that $rmp^{-m(p)}\equiv 1$ modulo $p^{m(p)}$, since $mp^{-m(p)}$ is itself prime to p. Hence $b\equiv b^{rmp^{-m(p)}}$ modulo F^*, and there exists therefore an element f in F^* such that $b=b^{rmp^{-m(p)}}f$. Since $b^{rm}=b^{p^{m(p)}}f^{-p^{m(p)}}$ is in F^*, it is in K_m^m and therefore in F^{*p}. Thus there exists an element c in F^* such that $c^p=b^{rm}$. If we put $d=cf^{p^{m(p)-1}}$, then d is an element in F^* and $(b^{p^{m(p)-1}}d^{-1})^p=b^{p^{m(p)}}f^{-p^{m(p)}}c^{-p}=1$. Hence $e=b^{p^{m(p)-1}}d^{-1}$ is a pth root of unity and as such is contained in F^*, proving finally that $b^{p^{m(p)-1}}=ed$ is an element in F^*. This completes the proof of the necessity of condition (b).

Suppose conversely that the conditions (a) and (b) are satisfied by K. If p is any prime number such that $m(p)=0$, then it is obvious

that K_m/F^* contains elements of order $p^{m(p)}$ $(=1)$. If $0<m(p)$, and if F^* does not contain any pth roots of unity different from 1, then we infer from (a) that K contains a primitive $p^{m(p)}$th root of unity e; and it is obvious that the order of e modulo F^* is exactly $p^{m(p)}$, proving again the existence of an element of order $p^{m(p)}$ in K_m/F^*. If finally $0<m(p)$, though F^* contains pth roots of unity different from 1, then we infer from condition (b) the existence of an element b in K_m^m which is not contained in F^{*p}. It is clear that b is an element in F^* and that there exists an element c in K satisfying $b=c^{p^{m(p)}}=(c^{p^{m(p)-1}})^p$. Since b is not in F^{*p}, it is impossible that $c^{p^{m(p)-1}}$ belongs to F^*, though $c^{p^{m(p)}}$ is in F^*, and this shows that c is an element in K whose order modulo F^* is exactly $p^{m(p)}$. Thus we have shown that K_m/F^* contains elements of order $p^{m(p)}$ for every p, and that consequently the maximum order of the elements in K_m/F^* is exactly m, as was to be proved.

REMARKS. 1. If $K=F(K_m)$, then condition (a) may be seen readily to be equivalent to the following condition:

(a′) *K is obtained by adjoining to F all the mth roots of the elements in the subgroup K_m^m of F^*.*

On the basis of Theorem 3 we may restrict ourselves without loss of generality to the consideration of extensions K of F which meet the requirement (a′).

2. Since the second root of unity -1 is always contained in F, it follows from $0<m(2)$ that the characteristic of F is different from 2, and that therefore condition (b) implies $K_m^m \subsetneq F^{*2}$.

3. Suppose that $0<m(p)$. It is well known that an automorphism of a cyclic group of order $p^{m(p)}$ possesses fixed elements different from 1 if, and only if, the order of the automorphism under consideration is a power of p. We note furthermore that a pth root of unity different from 1 is contained in F if, and only if, it is contained in K and is left invariant by every F-automorphism of K. Thus it follows that the condition (b) is equivalent to the following condition (b′), provided (a) is satisfied by K.

(b′) *If $0<m(p)$, and if the group of automorphisms induced by F-automorphisms of K in the group of the $p^{m(p)}$th roots of unity in K is of order a power of p, then $K_m^m \subsetneq F^{*p}$.*

For a further analysis it will be necessary to translate the fundamental concepts of the theory of crossed characters into concepts from the theory of extensions of fields. We denote by $E(p)$ the group of the $p^{m(p)}$th roots of unity in K so that E is the direct product of its cyclic subgroups $E(p)$; we note that condition (a) of Theorem 3 is equivalent to the assertion that $E(p)$ is of order $p^{m(p)}$ for every p.

As we denoted by C the homomorphism mapping the F-automorphism g of K upon the automorphism g of E which it induces in E, so we denote by Cp the homomorphism of G which maps g in G upon the automorphism g which it induces in $E(p)$. The groups G_C and G_{Cp} consist of all those elements in G which induce in E and $E(p)$ respectively the identity automorphism, and we find that G_C is the group of $F(E)$-automorphisms of K, and that G_{Cp} is the group of $F(E(p))$-automorphisms of K. The subgroup $H(p)$ of G has been defined[14] as the group of all those elements in G which are mapped upon 1 by every Cp-character of G, and it is an immediate consequence of Theorem 1 that $H(p)$ is the group of all the $F(K_{p^{m(p)}})$-automorphisms of K.

THEOREM 4. *There exists one and only one multiplicative group S between F^{*m} and F^* such that K may be obtained by adjoining all the mth roots of elements in S to F if, and only if, the following conditions are satisfied by K:*

(i) *K contains m distinct mth roots of unity, and $K=F(K_m)$.*

(ii) *If the element g in G_{Cp} is of order p modulo $H(p)$, then it is contained in $H(p)G_C$.*

(iii) *If $1<m(2)$, and if there exists an F-automorphism of K which maps every element in $E(2)$ upon its inverse, then $G/(H(2)G_{C2}^2)$ is not commutative.*

PROOF. It is readily verified that condition (i) is necessary and sufficient for the existence of at least one group S between F^{*m} and F^* such that K may be obtained by adjoining all the mth roots of elements in S to F (take $S=K_m^m$). If (i) is satisfied, then there exists one and only one group S with the required property if, and only if, K_m is the only group T between F^* and K_m such that $K=F(T)$ and $E\subseteq T$. It is an immediate consequence of Theorem 2 and of Corollary 2 to Theorem 2 that K_m is the only group T between F^*E and K_m satisfying $K=F(T)$ if, and only if, the group of all the C-characters of G is the only strictly complete group of C-characters of G; and that this later property is equivalent to conditions (ii) and (iii) is an immediate consequence of Baer [2], Corollary IV.4.2.

On account of the considerations centered around Theorem 3 we define: *the field K is an m-extension of its subfield F if* (1) *K is finite, normal, separable over F*; (2) *K contains m distinct mth roots of unity*; (3) $K=F(K_m)$. We note that (2) implies in particular that the characteristic of K is either 0 or prime to m.

If K is an m-extension of F, then K may be obtained by adjoining to F all the mth roots of elements in the multiplicative group K_m^m be-

[14] Baer [2] I.3; for a more detailed analysis of $H(p)$, cp. Baer [2] III 2.

tween F^{*m} and F^*; and it is obvious that K is uniquely determined, up to equivalence, by m and K_m^m.

THEOREM 5. *If S is a multiplicative group between F^{*m} and F^*, then the following conditions are necessary and sufficient for obtaining an m-extension of F by adjoining to F all*[15] *the mth roots of elements in S.*

(i) *The characteristic of F is either 0 or prime to m.*

(ii) *S/F^{*m} is a finite group.*

PROOF. The necessity of (i) has been pointed out before. To show the necessity of (ii), we consider an m-extension K of F which may be generated by adjoining the mth roots of elements in S to F. Then $K=F(K_m)$ and $F^{*m}\subseteq S\subseteq K_m^m\subseteq F^*$. It is a consequence of the finiteness of K over F and of Theorem 1 that K_m/F^* is a finite group. A homomorphism of K_m/F^* upon K_m^m/F^{*m} is effected by mapping the element x in K_m upon the element x^m in K_m^m, proving the finiteness of K_m^m/F^{*m}, and this implies the necessity of condition (ii).

The sufficiency of the conditions (i) and (ii) is readily verified.

Our theory would be complete if we could prove that different groups S between F^{*m} and F^*, meeting the above requirements (i), (ii), lead to essentially different m-extensions. This, however, cannot be expected, as may be seen from Theorem 4.

We are now going to impose two conditions upon the Galois group of the equation $x^m-1=0$ in the field F. This equation is supposed to be separable, a property that may be expressed in two equivalent ways by saying either that the characteristic of F, if not 0, should be prime to m, or that the equation has m distinct roots in a suitable extension of F. Properties of the Galois group of $x^m-1=0$ are best described by using some finite, normal, and separable extension of F which contains m distinct mth roots of unity, as the Galois group of the equation is independent of the particular choice of this extension. Thus we denote by E the group of the m distinct mth roots of unity contained in some finite, normal, separable extension H of F (for example, $H=F(E)$), and by $E(p)$ the subgroup of E consisting of the $p^{m(p)}$th roots of unity. Now we may state our two requirements, as follows:

(A) *If $1<m(2)$, then none of the automorphisms of the group E, contained in the Galois group of $x^m-1=0$ over F, maps every element in $E(2)$ upon its inverse.*

(B) *If the automorphisms of the Galois group of $x^m-1=0$ over F induce in $E(p)$ a group of order a power of p, and if $E(p)\neq 1$, then*

[15] This is best understood by restricting one's attention to subfields of some fixed algebraically closed extension of the field F.

the order of every automorphism in the Galois group of $x^m-1=0$ over F which leaves the elements in $E(p)$ invariant is prime to p.

It should be noted that condition (A) concerns only the Galois group of $x^{2^{m(2)}}-1=0$ over F, whereas (B) may be restated without reference to the groups E, $E(p)$ as follows:

(B′) *If the Galois group of $x^{p^{m(p)}}-1=0$ over F is of order a power of p, and if $0<m(p)$, then the modulo $x^{p^{m(p)}}-1=0$ reduced Galois group of $x^m-1=0$ over F is of an order prime to p.*

Witt in his theory of abelian extensions[16] had to require that m distinct mth roots of unity are contained in F. If we substitute for this condition the above properties (A), (B), then we obtain an extension of Witt's theory which comprises certain classes of m-extensions, as may be seen from the next theorem together with our remarks in connection with Theorem 5.

THEOREM 6. *If conditions* (A) *and* (B) *are satisfied by the field F and the integer m (which is prime to the characteristic of F in case the characteristic of F should be different from* 0), *if S and T are multiplicative groups between F^{*m} and F^*, and if S/F^{*m} and T/F^{*m} are both finite groups, then the following two properties imply each other*:

(i) $S=T$.

(ii) *The extensions of F which are obtained by adjoining to F all the mth roots of elements in S and in T respectively are equivalent.*

PROOF. It has been pointed out before that (ii) is a consequence of (i). If conversely (ii) is satisfied by S and T, then there exists a finite, normal, separable extension K of F such that $K=F(S')=F(T')$ where X' (for $X=S$ or T) is the group of all the elements in K whose mth powers are in X and X' consists of all the mth roots of elements in X. Clearly X is a group between F^* and K_m which contains m distinct mth roots of unity. It is a consequence of Theorem 2 and of Corollary 2 that X'/F^* is (essentially) a strictly complete group of C-characters of G. It is a consequence of conditions (A), (B) and of Baer [2], Theorem IV.1.1, that the only strictly complete group of C-characters of G is the group of all the C-characters of G. But it follows from Theorem 1 that this latter group is essentially the same as K_m/F^*, proving that $X'=K_m$. Thus we have shown that $S'=T'=K_m$, and this implies clearly $S=T$.

REMARK. If m is in particular an odd prime power, then conditions (A) and (B) are satisfied by F and m; if m is a power of 2, then (B) is satisfied by F and m, and (A) would be satisfied, for example, if F

[16] Witt [1].

contained all the fourth roots of unity. In these cases the above theorem may be applied without reservation.

In case condition (B) is not satisfied by F and m, we have to analyze the situation still further in order to obtain an extension of Theorem 6.

If K is an m-extension of F, then $K=F(K_m)$, and the multiplicative group K_m is the product of the multiplicative groups $K_{p^{m(p)}}$ consisting of all the elements in K whose $p^{m(p)}$th power is in F^*. Clearly K is the composite of its uniquely determined subfields $F(K_{p^{m(p)}})=K^{(p)}$, and it is readily verified that $K^{(p)}$ is a $p^{m(p)}$-extension of F. It has been remarked just now that condition (B) is satisfied by F and $p^{m(p)}$; and thus Theorem 6 may be applied upon $K^{(p)}$, provided (A) is satisfied by F and $2^{m(2)}$. Furthermore it should be remarked that a composite of $p^{m(p)}$-extensions of F is an m-extension of F. Combining these remarks with Theorem 5 we obtain the following result.

THEOREM 7. *Suppose that the characteristic of the field F is either* 0 *or prime to the integer m.*

(a) *The field K is an m-extension of the field F if, and only if, there exists, for every prime p, a multiplicative group S_p between $F^{*p^{m(p)}}$ and F^* such that $S_p/F^{*p^{m(p)}}$ is finite, and such that K is obtained by adjoining to F all the $p^{m(p)}$th roots of the elements in S_p.*

(b) *If condition* (A) *is satisfied by F and m, if S_p and T_p are, for every p, multiplicative groups between $F^{*p^{m(p)}}$ and F^* such that both $S_p/F^{*p^{m(p)}}$ and $T_p/F^{*p^{m(p)}}$ are finite, then the following two properties are equivalent:*

(b, i) *$S_p=T_p$ for every prime p.*

(b, ii) *The m-extensions of F which are determined by the S_p and by the T_p respectively through statement* (a) *are equivalent extensions of F.*

If, however, condition (A) fails to hold, then we have to fall back upon Theorem 4.

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A NOTE ON DIFFERENTIAL POLYNOMIALS

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The following theorem indicates to what extent the expression of a differential polynomial[1] G as an element of the differential ideal determined by F is unique.

THEOREM I. *Let $F \neq 0$, $C_0, C_1, \cdots, C_s$ be differential polynomials in the unknowns $y_1, \cdots, y_n$ with coefficients in an abstract differential field $\mathfrak{J}$. Let $F^{(i)}$ be the ith derivative of F and let*

$$C_0F + C_1F' + \cdots + C_sF^{(s)} \tag{1}$$

be identically zero. Then each C_i is in the perfect ideal generated by F.[2]

We need merely show that any solution $y_j = \bar{y}_j$ $(j = 1, \cdots, n)$, in any extension $\mathfrak{J}_1$ of $\mathfrak{J}$, of the form F is a solution of each C_i.[3] Since this is true if F has no solutions, we may assume that F effectively involves the unknowns. Make the substitution $y_j = z_j + \bar{y}_j$ in (1). Let A consist of the terms of F of lowest degree in the z_j and their derivatives. Collecting terms of the same degree, we see that

$$C_0(\bar{y})A + \cdots + C_s(\bar{y})A^{(s)} = 0, \tag{2}$$

where $C_i(\bar{y})$ is the element of $\mathfrak{J}_1$ obtained by substituting $y_j = \bar{y}_j$ $(j = 1, \cdots, n)$ in C_i. Let A be of order $p \geqq 0$ in some z_k which it effectively involves, let $z_{k,m}$ be the mth derivative of z_k, and let S be the partial derivative of A with respect to $z_{k,p}$. For $i > 0$, $A^{(i)}$ can be written as $Sz_{k,p+i} + B_i$, where B_i is some form of order less than $p+i$ in z_k. Now (2) becomes

$$C_s(\bar{y})Sz_{k,p+s} + D = 0 \tag{3}$$

where D has order less than $p+s$ in z_k. Hence $C_s(\bar{y}) = 0$. In turn $C_{s-1}, \cdots, C_0$ must vanish for $y_j = \bar{y}_j$ as desired.

Using the ideas of the above proof together with a uniqueness result of J. F. Ritt,[4] one can very easily prove the following generalization.

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[1] For definitions of differential fields, polynomials, and ideals, see H. W. Raudenbush, Ann. of Math. (2) vol. 34 (1933) pp. 509–517.

[2] For a result analogous to Theorem I for ordinary polynomials, see Satz 1 of E. Lasker, *Zur Theorie der Moduln und Ideale*, Math. Ann. vol. 60 (1905) pp. 20–116.

[3] H. W. Raudenbush, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 361–368.

[4] *On singular solutions* · · · , Ann. of Math. vol. 37 (1936) pp. 552–617, §§1–3.

THEOREM II. *Let $C_1P_1+\cdots+C_sP_s$ be identically zero, where the P_i are distinct power products each of degree $d>0$ in a nonzero F and its derivatives. Then each C_i is in the perfect ideal generated by F.*

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ON THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF FORM $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$

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One of the oldest unsolved mathematical problems is the following one: Are there odd perfect numbers?[2] If such a number n exists, it must have the form

$$n = p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}$$

where $p, q_1, q_2, \cdots, q_t$ are primes and $p\equiv\alpha\equiv 1 \pmod 4$. This has been proved by Euler.[3] Sylvester[4] obtained estimates for t, in particular $t\geqq 4$, and $t\geqq 7$ if $n\not\equiv 0 \pmod 3$. Recently, it was shown by R. Steuerwald[5] that the case $\beta_1=\beta_2=\cdots=\beta_t=1$ is impossible, and by H. J. Kanold[6] that the same is true for $\beta_1=\beta_2=\cdots=\beta_t=2$. Moreover Kanold proved that n is not perfect if the greatest common divisor d of $2\beta_1+1, 2\beta_2+1, \cdots, 2\beta_t+1$ is divisible by 9, 15, 21, or 33, and some similar results. All these results deal with the case $d>1$.

In the following, it will be proved that no odd perfect number n of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$ exists. Here we have $d=1$. For the proof I use

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[1] The relation of the results of this paper to another paper by H. J. Kanold, *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math. vol. 184 (1942) pp. 116–124, will be considered in an addendum to be published in the December Bulletin.

[2] For the history of the problem see Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33.

[3] *Commentationes arithmeticae collectae*, vol. 2, *Tractatus de numerorum doctrina* 1849, p. 514; *Opera postuma*, vol. 1, 1862, pp. 14–15.

[4] *Sur l'impossibilité de l'existence d'un nombre parfait impair qui ne contient pas au moins 5 diviseurs premiers distincts*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 522–526; *Collected mathematical papers*, vol. 4, 1912, pp. 611–614. *Sur une classe spéciale des diviseurs de la somme d'une série géométrique*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 446–450; *Collected mathematical papers*, vol. 4, 1912, pp. 607–610.

[5] *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München, 1937, pp. 68–72.

[6] *Untersuchungen über ungerade vollkommene Zahlen*, J. Reine Angew. Math. vol. 183 (1941) pp. 98–109.

theorems of T. Nagell on Diophantine equations. With the same method similar results may be obtained.

LEMMA 1. *Let q be a positive prime. The Diophantine equation*

$$q^2 + q + 1 = y^m$$

has no solution for $m > 1$.

PROOF. T. Nagell[7] has proved the following theorem: *If $m > 1$ is not a power of 3, then the Diophantine equation $x^2+x+1=y^m$ has no solutions in integers x, y with $y \neq \pm 1$.* In order to obtain all the solutions of

$$x^2 + x + 1 = y^3, \tag{1}$$

it is sufficient to solve the cubic Diophantine equation

$$a^3 - 3ab^2 + b^3 = 1 \tag{2}$$

and to set

$$x = a^3 - 3a^2b + b^3 - 1 \quad \text{and} \quad x = -a^3 + 3a^2b - b^3. \tag{3}$$

It follows from the Theorem of Thue-Siegel that (2) has only a finite number of solutions. Nagell gives the solutions

$$a=1,\ b=0;\quad a=0,\ b=1;\quad a=b=-1;\quad a=2,\ b=-1;\quad a=1,\ b=3. \tag{4}$$

But it is unknown whether there are other solutions in integers a, b. Since the discriminant of (2) is positive, we can not apply the general theorems of Delaunay and Nagell for cubic Diophantine equations. It follows from (3) and (4) that (1) has at least the following solutions: $x=0$, $y=1$; $x=-1$, $y=1$; $x=18$, $y=7$; $x=-19$, $y=7$.

By Nagell's theorem we have only to consider the case $m=3^k$ for the proof of our lemma. But we do not need the complete solution of (1); it is sufficient to prove that this equation has no solution where $x=q$ is a positive prime.

For $q=3$ we have $q^2+q+1=13$. This is no cube. If $q \equiv 1 \pmod 3$, then

$$q^2 + q + 1 \equiv 0 \pmod 3, \tag{5}$$

but

$$q^2 + q + 1 \not\equiv 0 \pmod{3^2} \tag{6}$$

[7] *Des équations indéterminées $x^3+x+1=y^n$ et $x^3+x+1=3y^n$*, Norsk Matematisk Forenings, Skrifter (1) no. 2 (1921). Cf. *L'analyse indéterminée de degré supérieure*, Mémorial des Sciences Mathématiques, vol. 39 (1929), p. 58.

since it is well known that the prime divisors of the pth cyclotomic polynomial $f_p(x)$, where p is a prime, are the primes of form $ph+1$ and p itself; but $f_p(x)$ is not divisible by p^2 for any integer x. It follows from (5) and (6) that q^2+q+1 is not a cube for $q\equiv 1 \pmod 3$.

Now, let q be a prime of form $3h+2$. Since $y^2+y+1<y^3$ for $y\geqq 2$, it follows from $q^2+q+1=y^3$ that $q>y$. Moreover we have $q(q+1)=(y-1)(y^2+y+1)$. Since q is a prime and greater than y, it follows that $y-1$ is relatively prime to q. Hence q is a divisor of y^2+y+1. This gives a contradiction because y^2+y+1 has no prime divisor of form $3h+2$.

LEMMA 2. *Let r and s be different positive integers and p be a prime. The system of simultaneous Diophantine equations $x^2+x+1=3p^r$, $y^2+y+1=3p^s$, has no solutions in positive integers x, y.*

PROOF. Nagell[8] has proved that the Diophantine equation $x^2+x+1=3z^k$ $(k>2)$ has no solution with $z>1$. Hence we have only to consider the case $r=1$, $s=2$. Then we have

$$x^2 + x + 1 = 3p, \qquad y^2 + y + 1 = 3p^2. \tag{7}$$

If these equations have a solution in positive integers, then it follows from (7) that

$$0 < x < p < y < 2p \tag{8}$$

and, on the other hand, that

$$(2x + 1)^2 \equiv -3 \equiv (2y + 1)^2 \pmod p,$$
$$2x + 1 \equiv \pm (2y + 1) \pmod p$$

since p is a prime, hence either

$$x \equiv y \pmod p \tag{9}$$

or

$$x \equiv -y - 1 \pmod p. \tag{10}$$

In the first case, it follows from (8) and (9) that

$$y = p + x \tag{11}$$

and in the second case, from (8) and (10) that

$$y = 2p - x - 1. \tag{12}$$

On the other hand, it follows from (7) that $p\neq 3$ and that $x\equiv y\equiv 1$

[8] Loc. cit.

(mod 3). This contradicts (11) and (12).

Now we are able to prove our theorem.

THEOREM. *An odd number of form* $n=p^{\alpha}q_1^2q_2^2\cdots q_{l-1}^2q_l^4$ *is not perfect.*

PROOF. We change the notation and write n in the following form

$$n = p^{\alpha}q_1^2q_2^2\cdots q_k^2r_1^2r_2^2\cdots r_l^2s^4 \qquad (k \geqq 0, l \geqq 0)$$

where the primes q_κ are congruent to 1 (mod 3) and where the primes r_λ are incongruent to 1 (mod 3). Let us assume that n is perfect, then we have

$$2n = \sigma(n) = \sigma(p^{\alpha})\sigma(q_1^2)\sigma(q_2^2)\cdots\sigma(q_k^2)\sigma(r_1^2)\sigma(r_2^2)\cdots\sigma(r_l^2)\sigma(s^4),$$

where $\sigma(n)$ denotes the sum of the divisors of n. It follows that

$$\begin{aligned} 2n &= 2p^{\alpha}q_1^2q_2^2\cdots q_k^2r_1^2r_2^2\cdots r_l^2s^4 \\ &= \sigma(p^{\alpha})\prod_{\kappa=1}^{k}(1+q_\kappa+q_\kappa^2)\prod_{\lambda=1}^{l}\{(1+r_\lambda+r_\lambda^2)\}(1+s+s^2+s^3+s^4). \end{aligned} \tag{13}$$

Each factor $1+q_\kappa+q_\kappa^2$ is divisible by 3, but not by 9; each factor $1+r_\lambda+r_\lambda^2$ is not divisible by 3. All the other prime divisors of

$$\prod_{\kappa=1}^{k}(1+q_\kappa+q_\kappa^2)\prod_{\lambda=1}^{l}(1+r_\lambda+r_\lambda^2)$$

have the form $3h+1$. We have now to distinguish between some cases.

I. $n\not\equiv 0$ (mod 3). Here we have $k=0$. Since n is not divisible by 3, it follows from Sylvester's theorem mentioned above that n must contain at least 8 different primes; hence $l\geqq 6$. Moreover we obtain from (13) that $\prod_{\lambda=1}^{l}(1+r_\lambda+r_\lambda^2)$ is a divisor of $p^{\alpha}s^4$. It may happen that one of the l factors of this product equals p, but each of the $l-1$ remaining factors cannot be a power of p by Lemma 1. Hence each of these $l-1$ factors must be divisible by s, and their product must be divisible by s^5. This gives a contradiction.

II. $n\equiv 0$ (mod 3), $n\not\equiv 0$ (mod 27). Here $k\leqq 2$. One of the primes r_λ, say r_1, equals 3 since $p=3$ is not possible. It follows that n is divisible exactly by 3^2, and hence by $\sigma(3^2)=13$, because of (13). Therefore we have either $13=p$, $13=q_1$, or $13=s$.

IIa. $p=13$. Since $\alpha+1$ is even, $\sigma(p^{\alpha})=(p^{\alpha+1}-1)/(p-1)$ is divisible by $p+1$. Hence n must be divisible by 7, and it follows that either one of the primes q_κ, say $q_1=7$, or $s=7$.

IIaα. $q_1=7$. n must be divisible by $\sigma(q_1^2)=57$, therefore by 19, and we have either $q_2=19$ or $s=19$.

IIaα1. $q_2=19$. n is divisible by $\sigma(q_2^2)=381$, hence by 127. Since $127\equiv 1 \pmod 3$ and $k\leqq 2$, we have $127=s$. Thus

$$n = 13^{\alpha}\cdot 3^2\cdot 7^2\cdot 19^2\cdot 127^4 r_2^2 r_3^2\cdots r_l^2,$$

$$1 = \frac{\sigma(n)}{2n} = \frac{\sigma(13^{\alpha})\cdot 13\cdot 57\cdot 381\cdot\sigma(127^4)\sigma(r_2^2)\cdots\sigma(r_l^2)}{2\cdot 13^{\alpha}\cdot 9\cdot 49\cdot 361\cdot 127^4\cdot r_2^2\cdots r_l^2} \tag{14}$$

$$= \frac{\{(1/14)\cdot\sigma(13^{\alpha})\}\sigma(127^4)\sigma(r_2^2)\cdots\sigma(r_l^2)}{13^{\alpha-1}\cdot 7\cdot 19\cdot 127^3\cdot r_2^2\cdots r_l^2}.$$

Since the prime divisors of $\sigma(127^4)$ have the form $5h+1$, they are different from 7, 13, 19, and 127. It follows now from (14) that they are of form $3h+2$, hence of form $15h+11$. Since

$$\sigma(127^4)\equiv 1+7+4+13+1\equiv 11 \pmod{15},$$

the number of prime divisors of $\sigma(127^4)$ must be odd if n should be perfect.

Let us first assume that $\sigma(127^4)$ is a prime. Then we have $\sigma(127^4)=r_\lambda$,

$$\sigma(r_\lambda^2) = 1+r_\lambda+r_\lambda^2 = 1+1+127+127^2+127^3+127^4 + (1+127+127^2+127^3+127^4)^2. \tag{15}$$

Setting $\sigma(r_\lambda^2)=A$ it follows from (14) that A can have only the prime divisors 7, 13, 19, and 127. But, by (15),

$$A\not\equiv 0 \pmod 7, \qquad A\not\equiv 0 \pmod{19}, \qquad A\not\equiv 0 \pmod{127}.$$

Since $A>13$, it follows that A is a power of 13. This contradicts Lemma 1.

Let us assume now that $\sigma(127^4)$ is composite. Since the number of its prime divisors would be odd, there would be at least 3 factors r_λ, and at least one of them, say r_2, must be less than $\{\sigma(127^4)\}^{1/3}$. But

$$\{\sigma(127^4)\}^{1/3} < \{127^3(127+1+.01)\}^{1/3} < 641,$$

therefore r_2 would be one of the following primes of form $15h+11$: 11, 41, 71, 101, 131, 191, 251, 281, 311, 401, 431, 461, 491, 521. Since $\sigma(r_2^2)\neq 7, 13, 19, 127$, it must be divisible by at least two of these primes, by Lemma 1. But $f(x)=1+x+x^2\equiv 0 \pmod 7$ for $x\equiv 2, 4$; $f(x)\equiv 0 \pmod{13}$ for $x\equiv 3, 9$; $f(x)\equiv 0 \pmod{19}$ for $x\equiv 7, 11$. Hence

it is easy to see that $\sigma(r_2^2)$ is relatively prime to $7\cdot 13\cdot 19$ for $r_2=41$, 71, 101, 131, 251, 281, 461, 491, 521. For $r_2=401$ and 431 we have $(\sigma(r_2^2), 13\cdot 19)=1$ and $\sigma(r_2^2)\not\equiv 0 \pmod{127}$. For the remaining primes 11 and 191 we have $\sigma(127^4)\not\equiv 0 \pmod{11}$ and $\sigma(127^4)\not\equiv 0 \pmod{191}$. This is impossible since r_2 was a divisor of $\sigma(127^4)$.

IIaα2. $s=19$. Since $\sigma(19^4)$ is divisible by 151, we have $q_2=151$, and since $\sigma(151^2)=22953=3\cdot 7\cdot 1093$, we have $q_3=1093$. This contradicts $k\leqq 2$.

IIaβ. $s=7$. Since $\sigma(7^4)=2801$ is a prime and since $\sigma(2801^2)=37\cdot 43\cdot 4933$, it follows that $q_1=37$, $q_2=43$, $q_3=4933$. This is impossible.

IIb. $q_1=13$. Since $\sigma(q_1^2)=3\cdot 61$, we have either p, q_2 or $s=61$.

IIbα. $p=61$. Here n is divisible by $(p+1)/2$, hence either $q_2=31$ or $s=31$.

IIbα1. $q_2=31$. Since $\sigma(q_2^2)=3\cdot 331$ and $k\leqq 2$, we have $s=331$. Therefore n is divisible by $\sigma(331^4)$, hence by 5, and n has the form

$$n = 61^{\alpha}\cdot 3^2\cdot 13^2\cdot 31^2\cdot 5^2\cdot 331^4\cdot M^2. \tag{16}$$

Since $\sigma(p^{\alpha})/p^{\alpha}\geqq(p^{\alpha}+p^{\alpha-1})/p^{\alpha}=(p+1)/p$, it follows from (16) that

$$\sigma(n)/2n > (62\cdot 13\cdot 183\cdot 993\cdot 31)/(2\cdot 61\cdot 9\cdot 13^2\cdot 31^2\cdot 25) = 331/325 > 1.$$

This is impossible.

IIbα2. $s=31$. $\sigma(31^4)\equiv 0 \pmod{11}$ and $\sigma(11^2)=7\cdot 19$. Hence $q_2=7$ and $q_3=19$. This contradicts $k\leqq 2$.

IIbβ. $q_2=61$. Since $\sigma(61^2)=3\cdot 13\cdot 97$, we have either $p=97$ or $s=97$. If $p=97$, then n is divisible by $(p+1)/2=7^2$, therefore $s=7$ since $q_1=13$, $q_2=61$, and $k\leqq 2$. But it was proved in IIaβ that $s=7$ is impossible. For $s=97$ the proof is the same as in IIbα2 since $97\equiv 31 \pmod{11}$.

IIbγ. $s=61$. $\sigma(61^4)$ is divisible by 5 and by 131. Therefore $r_2=131$, and either $p=5$ or $r_3=5$. If $p=5$, then $\sigma(p^{\alpha})$ contains the factor $p+1=6$. Thus $\sigma(n)$ can contain only one other factor 3, hence $k=1$. Since $\sigma(131^2)\not\equiv 0 \pmod{13}$ and $\sigma(131^2)\not\equiv 0 \pmod{61}$, it is necessary that $\sigma(131^2)$ and n contain another prime factor of form $3h+1$. This is impossible. If $r_3=5$, then n is divisible by $\sigma(5^2)=31$, hence $q_2=31$, $q_3=331$. This contradicts $k\leqq 2$.

IIc. $s=13$. Since $\sigma(13^4)=30941$ is a prime, we have either $p=30941$ or $r_2=30941$. But $p=30941$ is impossible because n must be divisible by $(p+1)/2$ and $30941\equiv -1 \pmod{27}$. Hence $n\equiv 0 \pmod{27}$; this contradicts $k\leqq 2$. Therefore $r_2=30941$, and n is divisible by $\sigma(30941^2)$.

Let us first assume that $\sigma(30941^2)$ is a prime, then $\sigma(30941^2)=q_1$ since $\sigma(30941^2)\equiv 3 \pmod 4$. Now $\sigma(q_1^2)\equiv 0 \pmod{151}$, therefore

$q_2=151$, and n is divisible by $\sigma(151^2)$, hence by 7. This contradicts $k\leqq 2$.

Let us now assume that $\sigma(30941^2)$ is composite. It is easy to see that all the prime factors of $\sigma(30941^2)$ are greater than 151. Since these prime factors are congruent to 1 (mod 3), it follows that either $q_1>150$ and $q_2>150$, or $q_1>150$ and $p>150$.

In the first case we have $k=2$ and hence $p\not\equiv 2 \pmod 3$, therefore $p\equiv 1 \pmod{12}$. Since $(p+1)/2$ is a divisor of n and since 19 cannot be a divisor of n, the case $p=37$ is impossible. It follows that $p>60$.

In the latter case we have $q_2>60$ if $k=2$, because otherwise $\sigma(q_2^2)$ is divisible by one of the primes 19, 127, 331, 7, or 631; this is impossible since these primes are not divisors of $\sigma(30941^2)$. Hence, if $k=2$, two of the primes q_1, q_2, p are greater than 150 and the third is greater than 60. The product

$$P=\prod_{\varkappa=1}^{2}\sigma(q_\varkappa^2)\prod_{\lambda=1}^{l}\sigma(r_\lambda^2)$$

must be a divisor of $9p^\alpha q_1^2q_2^2s^4$. Each factor $\sigma(q_\varkappa^2)$ is divisible by 3. But only one of them may have the form $3p^t$ by Lemma 2. Therefore at least one of the 8 prime factors of $q_1^2q_2^2s^4$ is a divisor of $\sigma(q_1^2)\sigma(q_2^2)$; hence only 7 of these factors may be divisors of the product $\sigma(r_1^2)\sigma(r_2^2)\cdots\sigma(r_l^2)$. It may happen that one of these l factors equals p; each of the others is not a power of p, by Lemma 1, and contains, therefore, at least one of the remaining 7 prime factors of $q_1^2q_2^2s^4$. It follows that $l\leqq 8$.

We have $r_1=3$; $r_2=30941$. The primes r_λ are different from 5, 11, 23, 47, 53, 83, since $q_1>60$ and $q_2>60$. Therefore $r_3\geqq 17$, $r_4\geqq 29$, $r_5\geqq 41$, $r_6\geqq 59$, $r_7\geqq 71$, $r_8\geqq 89$ and

$$\frac{\sigma(n)}{2n}<\frac{\sigma(151^2)\cdot 151\cdot 61\cdot 13\cdot 30941\cdot\sigma(17^2)\cdot\sigma(29^2)\cdot 41\cdot 59\cdot 71\cdot 89\cdot 30941}{2\cdot 151^2\cdot 150\cdot 60\cdot 9\cdot 13^4\cdot 17^2\cdot 29^2\cdot 40\cdot 58\cdot 70\cdot 88\cdot 30940}<1.$$

This is not possible. For $k=1$, it follows similarly that $l\leqq 7$, and the proof is the same.

III. $s=3$. Here we have $k\leqq 4$. Since n must be divisible by $\sigma(3^4)=121$, it follows successively that n is divisible by 7, 19, 127, 5419, and 31. These five primes are congruent to 1 (mod 3) and congruent to 3 (mod 4). Hence $k\geqq 5$; this is impossible. Herewith our theorem is proved for each case.

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TWO NOTES ON MEASURE THEORY

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I. In a recent paper [1],[1] Saks has indicated a construction whereby a Carathéodory outer measure can be produced on any compact metric space M, provided that a certain linear functional Φ is defined on the set $\mathfrak{C}$ of all continuous real-valued functions whose domain is M. The functional Φ is required to be non-negative for non-negative functions, and to have the property that if the sequence $\{f_n\}$ has the uniform limit 0, then the sequence $\Phi(f_n)$ is a null-sequence. (The measure itself can be defined without this last property.) The purpose of this note is to show that such a linear functional always exists, in a non-trivial form, specifically, so that $\Phi(1)=1$.

We consider the set $\mathfrak{C}$ as a linear space, and together with $\mathfrak{C}$ the linear space $\mathfrak{K}\subset\mathfrak{C}$, where $\mathfrak{K}$ consists of all constant functions. On the entire space $\mathfrak{C}$, we define a functional $p(f)=\sup_{x\in M} f(x)$. This least upper bound always exists, since M, being a compact metric space, is a bicompact space, on which every continuous real-valued function is bounded. It is easy to verify that $p(f+g)\leqq p(f)+p(g)$, for all $f, g\in\mathfrak{C}$, and that $p(tf)=tp(f)$ whenever t is a non-negative real number. We define a linear functional Φ on the subspace $\mathfrak{K}$ as follows: $\Phi(f)=f(x)$ for an arbitrary $x\in M$. It is clear that $\Phi(f)=p(f)$ for $f\in\mathfrak{K}$ and that Φ is linear on $\mathfrak{K}$. By virtue of the celebrated theorem of Hahn-Banach, it appears that Φ can be extended linearly to all of $\mathfrak{C}$ in such a fashion that $\Phi(f)\leqq p(f)$ for all $f\in\mathfrak{C}$. We further observe that Φ may be taken non-negative for non-negative functions. For, if Φ has been defined by the Hahn-Banach construction for all $f\in\mathfrak{B}$, where $\mathfrak{K}\subset\mathfrak{B}\subset\mathfrak{C}$, $\mathfrak{B}\neq\mathfrak{C}$, and if $g\in\mathfrak{C}-\mathfrak{B}$ and $g\geqq 0$, then the number $a=\inf_{f\in\mathfrak{B}}(p(f+g)-\Phi(f))$ is an upper bound to possible values for $\Phi(g)$. a, however, is plainly non-negative, so that $\Phi(g)$ may always be taken non-negative. Suppose now that the sequence of functions $\{f_n\}$ has the uniform limit 0. The function $\epsilon-f_n$ is non-negative for all $n>N(\epsilon)$, $N(\epsilon)$ being some natural number dependent upon the arbitrary positive number ϵ. Accordingly, $\Phi(\epsilon-f_n)=\Phi(\epsilon)-\Phi(f_n)=\epsilon\Phi(1)-\Phi(f_n)=\epsilon-\Phi(f_n)\geqq 0$. Likewise, it is easy to show that $\epsilon+\Phi(f_n)\geqq 0$ for all sufficiently large n. It follows at once that $\lim_{n\to\infty}\Phi(f_n)=0$. It is proved in Saks [1] that the functional Φ can

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[1] Numbers in brackets refer to correspondingly numbered articles in the bibliography at the end of the paper.

be used to define a Carathéodory outer measure under which every Borel set is measurable.

II. The present note has as its object the proof of the following result.

THEOREM. *If E is any infinite set, there exists a non-negative real-valued function Γ defined on all subsets of E such that*:

(1) $\Gamma(A\cup B)\leqq\Gamma(A)+\Gamma(B)$;
(2) $\Gamma(A)\leqq\Gamma(B)$ *if* $A\subset B$;
(3) $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$ *if* $A_n\cap A_m=0$ *for* $m\neq n$;
(4) $\Gamma(E)=1$;
(5) $\Gamma(0)=0$;
(6) *the function Γ assumes an infinite number of different values*;
(7) $\Gamma(\{p\})=0$ *for all points $p\in E$ except for those in a countable subset of E.*

Since Ulam has proved that a function enjoying properties (4) and (3) cannot vanish for all subsets containing exactly one point (where E has any of a wide class of cardinal numbers), it appears that the present theorem is the strongest result possible.

The proof of this theorem depends upon a consideration of the family $\mathfrak{B}$ of all bounded real-valued functions defined on the set E. As in the preceding note, it is easy to prove the existence of a linear functional Φ defined on the family $\mathfrak{B}$ considered as a linear space. The construction, for our present purposes, will be considered in more detail. Let E be partitioned into $\aleph_0$ disjoint sets $E_1, E_2, E_3, \cdots, E_n, \cdots$, each having cardinal number equal to the cardinal number of E. Let ω_n be the characteristic function of the set E_n. It is obvious that $\omega_n\in\mathfrak{B}$ for every n. We shall first define the linear functional Φ on the linear spaces $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \cdots, \mathfrak{P}_n, \cdots$ obtained from $\mathfrak{K}$, the space of constant functions, by adjoining $\omega_1, \omega_2, \omega_3, \cdots, \omega_n, \cdots$ in succession and forming all possible linear combinations. As in the preceding note, we define $p(f)$ as $\sup_{x\in E} f(x)$, and $\Phi(f)$ as $f(x)$ for $f\in\mathfrak{K}$, x being any point of E. By the Hahn-Banach construction, if $\Phi(f)$ is to be bounded by $p(f)$, we must have, when we calculate $\Phi(\omega_1)$, $a_1\leqq\Phi(\omega_1)\leqq b_1$, where $a_1=\sup_{f\in\mathfrak{K}}(-p(-f-\omega_1)-\Phi(f))$ and $b_1=\inf_{f\in\mathfrak{K}}(p(f+\omega_1)-\Phi(f))$. It is easy to show that $b_1=1$ and that $a_1=0$. We may, then, in accordance with the Hahn-Banach construction, take $\Phi(\omega_1)$ as $1/2$.

The numbers $a_2=\sup_{f\in\mathfrak{P}_1}(-p(-f-\omega_2)-\Phi(f))$ and $b_2=\inf_{f\in\mathfrak{P}_1}(p(f+\omega_2)-\Phi(f))$ are lower and upper bounds, respectively, for $\Phi(\omega_2)$. a_2 may be computed as 0, and b_2, as it is easy to see, is equal

to 1/2. We may thus put $\Phi(\omega_2)=1/4$. This process may be continued by finite induction; it is found that the function ω_n may be assigned the value $1/2^n$ under the functional Φ. Φ having been defined for the linear space generated by $\mathfrak{R}$ and the sequence $\{\omega_n\}$, the Hahn-Banach construction is carried out for the rest of $\mathfrak{B}$ in any fashion consonant with the restrictions of that theorem, provided that $\Phi(f)\geqq 0$ for non-negative functions f. We thus have a linear non-negative functional defined on all of the space $\mathfrak{B}$.

The measure $\Gamma(A)$ for every subset A of E can now be defined: $\Gamma(A)=\Phi(\omega_A)$, ω_A being the characteristic function of the set A. Properties (1)–(7) can now be established. It is plain that $\omega_{A\cup B}\leqq\omega_A+\omega_B$, whence $\Phi(\omega_A+\omega_B-\omega_{A\cup B})\geqq 0$, and consequently $\Phi(\omega_{A\cup B})\leqq\Phi(\omega_A)+\Phi(\omega_B)$, which inequality establishes (1). It is also obvious that $A\subset B$ implies that $\omega_A\leqq\omega_B$. From this, we infer property (2).

We examine (3) in some detail. If A and B are disjoint sets, it follows that $\omega_{A\cup B}=\omega_A+\omega_B$, and consequently $\Phi(\omega_{A\cup B})=\Phi(\omega_A)+\Phi(\omega_B)$, that is, the measure is additive for all subsets of E. We may thus state that all subsets of E are measurable in the sense of Carathéodory. It is easy to prove from this fact that if $\{A_n\}$ is any sequence of pairwise disjoint sets, then $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$. The proof may be carried over word for word from a similar proof in Saks [2, chap. 2, §4, p. 44, Theorem 4.1].

Statements (4), (5), and (6) are immediate consequences of the definitions of Φ and Γ.

To prove that Γ vanishes for all points except those in a countable subset, we assume the contrary. If an uncountable set T of points exist such that $\Gamma(\{p\})>0$ for every $p\in T$, then there is some $\epsilon>0$ with $\Gamma(\{p_n\})>\epsilon$, where $p_n\in T$, and $n=1, 2, 3, \cdots$. On account of property (3), we have $\Gamma(\sum_{n=1}^{\infty}\{p_n\})=\sum_{n=1}^{\infty}\Gamma(\{p_n\})=\infty$. Since $1=\Gamma(E)\geqq\Gamma(\sum_{n=1}^{\infty}\{p_n\})$, a contradiction is apparent.

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ON CERTAIN PAIRS OF SURFACES IN ORDINARY SPACE

BUCHIN SU

1. **Introduction.** In a recent paper[1] Jesse Douglas has proposed and solved the following problem: To determine the form of the linear element of a surface in ordinary space upon which exists a family of ∞^2 curves possessing two properties: (1) The angular excess of any triangle ABC formed by curves of the family $\mathcal{J}$ is proportional to the area of the triangle:

$$\mathcal{E} = A + B + C - \pi = k\mathcal{A}, \tag{1}$$

where k denotes a constant; (2) The curves of $\mathcal{J}$ are a *linear* system; that is, a point transformation exists which converts them into the straight lines of a plane. It is natural to inquire what class of surfaces we shall obtain if, instead of using property (2), we make the less specific demand that a point transformation exists which converts the curves of $\mathcal{J}$ into the *geodesics* of another surface. Here we have found certain pairs of surfaces S and S_1 which furnish the complete solution of our generalized problem. According to whether the constant k is zero or not, the linear elements of S and S_1 take different types, whose derivation constitutes the purpose of the present paper.

2. **Conditions for the property $\mathcal{E}=k\mathcal{A}$.** As was shown by Douglas,[2] the necessary and sufficient conditions that every curve of a family $\mathcal{J}$ upon a surface S should have the property $\mathcal{E}=k\mathcal{A}$ can be expressed by the relation

$$ds/\rho = Pdu + Qdv, \tag{2}$$

where $1/\rho$ is the geodesic curvature of the curve and P, Q obey the condition

$$Q_u - P_v = (k - K)W. \tag{3}$$

For the subsequent discussion it is convenient to consider both surfaces S and S_1, wherein the curves of $\mathcal{J}$ upon S correspond to the geodesics of S_1. Let (u, v) be general coordinates of the corresponding points on these surfaces, so that the first fundamental form of S is

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[1] J. Douglas, *A new special form of the linear element of a surface*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 101–116.

[2] Douglas, loc. cit., p. 108.

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{4}$$

and that of S_1 is

$$ds_1^2 = E_1du^2 + 2F_1dudv + G_1dv^2. \tag{5}$$

According to the classical theorem of Tissot,[3] there exists upon S one and, in general, only one orthogonal system of curves which corresponds to an orthogonal system upon S_1. Suppose that these surfaces S and S_1 are referred to the orthogonal curves which correspond to each other; then we have

$$ds^2 = Edu^2 + Gdv^2, \tag{6}$$

$$ds_1^2 = E_1du^2 + G_1dv^2. \tag{7}$$

In orthogonal coordinates (u, v), the geodesic curvature $1/\rho$ of any curve $v=v(u)$ upon the surface S is given by[4]

$$\begin{aligned} ds/\rho = {} & (EG)^{-1/2}(E+Gv'^2)^{-1}\{EGv'' - (1/2)EE_v \\ & + (EG_u - (1/2)GE_u)v' + ((1/2)EG_v - GE_v)v'^2 + (1/2)GG_uv'^3\}du. \end{aligned} \tag{8}$$

Therefore, for a family $\mathcal{F}$, we have by (2):

$$v'' = A + Bv' + Cv'^2 + Dv'^3, \tag{9}$$

where

$$\begin{cases} A = (1/2)E_v/G + (E/G)^{1/2}P, \\ B = (1/2)E_u/E - G_u/G + (E/G)^{1/2}Q, \\ C = E_v/E - (1/2)G_v/G + (G/E)^{1/2}P, \\ D = -(1/2)G_u/E + (G/E)^{1/2}Q, \end{cases} \tag{10}$$

and

$$Q_u - P_v = (k - K)(EG)^{1/2}. \tag{11}$$

That is: *the form* (9) *with additional condition* (11) *is characteristic of curves having the property* $\mathfrak{E}=k\mathcal{A}$ *upon the surface* S.

3. Geodesic representation of the family $\mathcal{F}$. We now have to impose the further property on the family $\mathcal{F}$.

Since the parametric curves on the surface S_1 form an orthogonal system, the differential equation of geodesics of S_1 is found to be[5]

[3] Cf. G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, 1894, p. 47.

[4] Cf. W. Blaschke, *Vorlesungen über Differentialgeometrie*, 3d edition, 1930, p. 175. Write $F=0$, $u'=1$, $u''=0$.

[5] Darboux, loc. cit., p. 49.

$$
\begin{aligned}
v'' = {} & \frac{1}{2}\frac{\partial E_1}{\partial v}\Big/G_1 + \left(\frac{1}{2}\frac{\partial E_1}{\partial u}\Big/E_1 - \frac{\partial G_1}{\partial u}\Big/G_1\right)v' \\
& + \left(\frac{\partial E_1}{\partial v}\Big/E_1 - \frac{1}{2}\frac{\partial G_1}{\partial v}\Big/G_1\right)v'^2 - \frac{1}{2}\left(\frac{\partial G_1}{\partial u}\Big/E_1\right)v'^3.
\end{aligned} \tag{12}
$$

In order that they should correspond to the curves of the family $\mathfrak{F}$ upon S, it is necessary and sufficient that the differential equations (9) and (12) be coincident with each other. This gives

$$
\begin{cases}
\dfrac{1}{2}\dfrac{\partial E}{\partial v}\Big/G + (E/G)^{1/2}P = \dfrac{1}{2}\dfrac{\partial E_1}{\partial v}\Big/G_1, \\
\dfrac{1}{2}\dfrac{\partial E}{\partial u}\Big/E - \dfrac{\partial G}{\partial u}\Big/G + (E/G)^{1/2}Q = \dfrac{1}{2}\dfrac{\partial E_1}{\partial u}\Big/E_1 - \dfrac{\partial G_1}{\partial u}\Big/G_1, \\
\dfrac{\partial E}{\partial v}\Big/E - \dfrac{1}{2}\dfrac{\partial G}{\partial v}\Big/G + (G/E)^{1/2}P = \dfrac{\partial E_1}{\partial v}\Big/E_1 - \dfrac{1}{2}\dfrac{\partial G_1}{\partial v}\Big/G_1, \\
\dfrac{1}{2}\dfrac{\partial G}{\partial u}\Big/E - (G/E)^{1/2}Q = \dfrac{1}{2}\dfrac{\partial G_1}{\partial u}\Big/E_1.
\end{cases} \tag{13}
$$

From the first and the last of these equations we have the expressions for P and Q:

$$
\begin{cases}
P = \dfrac{1}{2}(G/E)^{1/2}\left\{\dfrac{\partial E_1}{\partial v}\Big/G_1 - \dfrac{\partial E}{\partial v}\Big/G\right\}, \\
Q = \dfrac{1}{2}(E/G)^{1/2}\left\{\dfrac{\partial G}{\partial u}\Big/E - \dfrac{\partial G_1}{\partial u}\Big/E_1\right\}.
\end{cases} \tag{14}
$$

Substitution of these expressions in the remaining equations of (13) shows that the fundamental quantities E, G; E_1, G_1 are related by

$$
\begin{cases}
\dfrac{\partial}{\partial v}\log\dfrac{EG_1}{GE_1} = \left(1 - \dfrac{GE_1}{EG_1}\right)\dfrac{\partial \log E_1}{\partial v}, \\
\dfrac{\partial}{\partial u}\log\dfrac{EG_1}{GE_1} = \left(\dfrac{EG_1}{GE_1} - 1\right)\dfrac{\partial \log G_1}{\partial u}.
\end{cases} \tag{15}
$$

In interpreting these conditions, it is important to distinguish the case $(E/G):(E_1/G_1)=1$ from $(E/G):(E_1/G_1)\neq 1$. If $(E/G):(E_1/G_1)=1$, then we can put

$$
E_1 = \rho E, \qquad G_1 = \rho G, \tag{16}
$$

where $\rho \neq 0$, so that the surfaces S and S_1 are conformal. In this case the equations (15) are satisfied identically, while the equations (14) give

$$(17) \qquad \begin{cases} P = \dfrac{1}{2}(E/G)^{1/2}\dfrac{\partial \log \rho}{\partial v}, \\ Q = -\dfrac{1}{2}(G/E)^{1/2}\dfrac{\partial \log \rho}{\partial u}. \end{cases}$$

Substituting these expressions in (11) we have

$$(18) \qquad \frac{\partial}{\partial v}\left\{(E/G)^{1/2}\frac{\partial \log \rho}{\partial v}\right\} + \frac{\partial}{\partial u}\left\{(G/E)^{1/2}\frac{\partial \log \rho}{\partial u}\right\} + 2(k - K)(EG)^{1/2} = 0.$$

This condition can be interpreted geometrically as follows: by means of the formula of G. Frobenius for Gaussian curvature[6] we obtain for the surface S with linear element (6)

$$(19) \qquad K = -\frac{1}{2W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\},$$

where $W = (EG)^{1/2}$, and similarly, for the surface S_1 with linear element (7)

$$(20) \qquad K_1 = -\frac{1}{2W_1}\left\{\frac{\partial}{\partial v}\left(\frac{\partial E_1}{\partial v}\Big/ W_1\right) + \frac{\partial}{\partial u}\left(\frac{\partial G_1}{\partial u}\Big/ W_1\right)\right\},$$

where $W_1 = (E_1G_1)^{1/2}$.

If the expressions of E_1 and G_1 given by (16) are substituted in the right-hand member of (20), then

$$K_1 = -\frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\} - \frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}\left((E/G)^{1/2}\frac{\partial \log \rho}{\partial v}\right) + \frac{\partial}{\partial u}\left((G/E)^{1/2}\frac{\partial \log \rho}{\partial u}\right)\right\}.$$

A reference to (18) and (19) gives immediately the relation

$$(21) \qquad \rho K_1 = k,$$

which is, of course, equivalent to (18).

If $k \neq 0$, then we have

[6] Blaschke, loc. cit., p. 117.

$$(22)\qquad \begin{cases} ds^2 = (K_1/k)(E_1du^2 + G_1dv^2), \\ ds_1^2 = E_1du^2 + G_1dv^2. \end{cases}$$

Therefore *the surface S_1 may be arbitrarily selected, the only restriction being that it is non-developable, and any surface S, necessarily conformal to S_1, with the linear element*

$$(23)\qquad ds^2 = (K_1/k)ds_1^2$$

possesses the two stated properties, where K_1 is the Gaussian curvature of S_1 and k a constant different from zero. In particular when a surface of constant Gaussian curvature k is taken for S_1, the corresponding surface S is applicable to S_1.

This result not only proves the existence but also furnishes a remarkable class of the surfaces under consideration.

On the contrary, if $k=0$ in the relation (21), then K_1 must necessarily vanish, because ρ is by no means zero, so that *the surface S_1 is developable*. In this case, the surface S is arbitrary and $\mathfrak{J}$ must be a conformal image of the ∞^2 straight lines of a plane. That no other family of curves upon a generic surface S can be linear and such that the sum of the angles in every triangle of curves in the family is two right angles has been proved analytically by E. Kasner[7] and synthetically by J. Douglas.[8]

We now consider the case where

$$(24)\qquad (E/G):(E_1/G_1) \neq 1.$$

The partial differential equations (15) are easily integrated, and the result may be written in the form

$$(25)\qquad (EG_1)/(GE_1) = 1 - E_1U^2, \quad (GE_1)/(EG_1) = 1 + G_1V^2,$$

where U denotes any function of u alone, and V any function of v alone. The assumption (24) shows that neither U nor V is zero.

From (25) follows the relation

$$(26)\qquad E_1U^2(G_1V^2 + 2) = G_1V^2(2 - E_1U^2).$$

It may happen that both members of (26) are zero. Since E_1, G_1, U, V are different from zero, we have in this case

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[7] E. Kasner, *A characteristic property of isothermal systems of curves*, Math. Ann. vol. 59 (1904) pp. 352–354.

[8] J. Douglas, *A criterion for the conformal equivalence of a Riemann space to a Euclidean space*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 299–306.

$$E_1 = 2U^{-2}, \qquad G_1 = -2V^{-2}, \tag{27}$$

so that

$$E/G = V^2/U^2. \tag{28}$$

The expressions (14) now become

$$P = -\frac{1}{2}\frac{V}{U}\frac{\partial \log E}{\partial v}, \qquad Q = \frac{1}{2}\frac{U}{V}\frac{\partial \log G}{\partial u}. \tag{29}$$

Substituting them in (18) and remembering the formula (19) in addition to the relation (28), we obtain

$$k = 0. \tag{30}$$

That is, *the sum of the angles in every triangle formed by three curves of the family $\mathfrak{F}$ under consideration must be two right angles*, and the linear elements of the surfaces S and S_1 are, after a suitable transformation of the type

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v), \tag{31}$$

reducible to the form

$$\begin{cases} ds^2 = \lambda(u, v)(du^2 + dv^2), \\ ds_1^2 = du^2 - dv^2. \end{cases} \tag{32}$$

It remains for us to consider the case

$$(2 - E_1U^2)(G_1V^2 + 2) \neq 0. \tag{33}$$

Setting (26) in the form

$$(E_1U^2)/(2 - E_1U^2) = (G_1V^2)(G_1V^2 + 2) = \tau, \tag{34}$$

we obtain

$$E_1 = 2\tau(1 + \tau)^{-1}U^{-2}, \qquad G_1 = 2\tau(1 - \tau)^{-1}V^{-2} \tag{35}$$

and

$$(E/G)^{1/2} = (1 - \tau)(1 + \tau)^{-1}VU^{-1}. \tag{36}$$

The expressions (14) for P and Q may be written in the form

$$\begin{cases} P = \frac{1}{2}\left\{\frac{V}{U}\frac{\partial \log E_1}{\partial v} - \frac{1}{W}\frac{\partial E}{\partial v}\right\}, \\ Q = -\frac{1}{2}\left\{\frac{U}{V}\frac{\partial \log G_1}{\partial u} - \frac{1}{W}\frac{\partial G}{\partial u}\right\}. \end{cases} \tag{37}$$

Consequently, we have

$$Q_u - P_v = -\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) + \frac{1}{2}\left\{\frac{\partial}{\partial u}(G_u/W) + \frac{\partial}{\partial v}(E_v/W)\right\},$$

namely,

$$Q_u - P_v = -\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) - WK, \tag{38}$$

as may easily be seen on account of (19). By comparison of (38) with (11) it follows that

$$2k(EG)^{1/2} = -\frac{1}{V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right). \tag{39}$$

In virtue of (35) we reach the relation:

$$2k(EG)^{1/2} = -\frac{1}{UV}\left[\left(U\frac{\partial}{\partial u}\right)^2\log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2\log\frac{\tau}{1+\tau}\right]. \tag{40}$$

It is important to distinguish the case $k=0$ from $k\neq 0$. The latter case is of more interest; we find by means of (40) an additional relation between E and G, namely,

$$(EG)^{1/2} = -(1/2kUV)\left[\left(U\frac{\partial}{\partial u}\right)^2\log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2\log\frac{\tau}{1+\tau}\right]. \tag{41}$$

Therefore the linear elements of S and S_1 are, from (35), (36) and (41), given by

$$\begin{cases} ds^2 = -\dfrac{1}{2k}\left[\left(U\dfrac{\partial}{\partial u}\right)^2\log\dfrac{\tau}{1-\tau} + \left(V\dfrac{\partial}{\partial v}\right)^2\log\dfrac{\tau}{1+\tau}\right] \\ \qquad \cdot\left[\dfrac{1-\tau}{1+\tau}U^{-2}du^2 + \dfrac{1+\tau}{1-\tau}V^{-2}dv^2\right], \\ ds_1^2 = \dfrac{2\tau}{1+\tau}U^{-2}du^2 + \dfrac{2\tau}{1-\tau}V^{-2}dv^2, \end{cases} \tag{42}$$

where τ denotes an arbitrary function of u, v.

If the parameters u, v are subjected to a suitable transformation

of type (31), then the pair of surfaces S and S_1 is characterized by the linear elements of a special form:

$$(43)\quad \begin{cases} ds^2 = -\dfrac{1}{2k}\left\{\dfrac{\partial^2}{\partial u^2}\log\dfrac{\tau}{1-\tau}+\dfrac{\partial^2}{\partial v^2}\log\dfrac{\tau}{1+\tau}\right\}\left\{\dfrac{1-\tau}{1+\tau}\,du^2+\dfrac{1+\tau}{1-\tau}\,dv^2\right\}, \\ ds_1^2 = \dfrac{2\tau}{1+\tau}\,du^2+\dfrac{2\tau}{1-\tau}\,dv^2, \end{cases}$$

involving an arbitrary function τ of u and v.

If $k=0$, then (39) becomes, by applying a transformation of type (31),

$$(44)\quad \frac{\partial^2}{\partial v^2}\log E_1+\frac{\partial^2}{\partial u^2}\log G_1 = 0.$$

Therefore we obtain

$$(45)\quad \begin{cases} ds_1^2 = (E^{-1/2}-G^{-1/2})(E^{1/2}du^2+G^{1/2}dv^2), \\ ds^2 = Edu^2+Gdv^2, \end{cases}$$

where the quantities E and G are related by

$$(46)\quad \frac{\partial^2}{\partial v^2}\log\left(1-(E/G)^{1/2}\right)+\frac{\partial^2}{\partial u^2}\log\left((G/E)^{1/2}-1\right) = 0.$$

Thus the problem of determining the form of linear elements of a pair of surfaces S and S_1 with the stated properties is completely solved.

It should be observed that the analogous problem for higher dimensional spaces may be of some interest. We hope to consider it in a future paper.

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SUBSERIES OF A CONVERGENT SERIES

HARRY POLLARD

In a recent paper J. D. Hill[1] has discussed the mean-value of the subseries of any absolutely convergent series $s=\sum u_n$. Simplifying his method by use of the Rademacher functions, we obtain a mapping of the subseries into the interval $0 \leqq x \leqq 1$ by defining[2]

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1 + R_n(x)}{2} u_n. \tag{1}$$

Hill's result states that if $\sum |u_n|$ converges, then the mean-value is given by

$$\int_0^1 \phi(x)dx = s/2. \tag{2}$$

In the theorem below we point out the weakest condition on the series $\sum u_n$ for which this result persists.

LEMMA. *If* (1) *converges on a set of positive measure it converges almost everywhere.*

Let D be the set of points on which (1) converges. Let $x = a_1a_2a_3 \cdots$ (in binary notation) be a point of D. If a finite number of the a_i are changed then the new point still belongs to D, for by the definition of the Rademacher functions this operation changes only a finite number of the terms of the series (1). Then D is a "homogeneous" set not of measure 0; hence it must be of measure 1.[3]

THEOREM. *A necessary and sufficient condition that the series* (1) *converge on a set of positive measure is that the two series* $\sum u_n$ *and* $\sum u_n^2$ *converge. Then* (1) *converges almost everywhere and* (2) *is valid.*

(i) Suppose that (1) converges on a set of positive measure. Then it must, by the lemma, converge almost everywhere. Then there exist

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[1] Bull. Amer. Math. Soc. vol. 48 (1942) p. 103.

[2] The mapping is not 1-1 at the points $x=k/2^n$, but this does not affect the results. For the properties of the Rademacher functions used in this paper see Kacmarcz and Steinhaus, *Le système orthogonale de M. Rademacher*, Studia Mathematica vol. 2 (1939) p. 231.

[3] C. Visser, *The law of nought-or-one in the theory of probability*, Studia Mathematica vol. 7 (1938) pp. 146–147.

two points x_0 and $1-x_0$, symmetric in $x=1/2$, at which (1) converges. Inasmuch as $R_n(x_0)+R_n(1-x_0)=0$ we have from (1) that $\phi(x_0)+\phi(1-x_0)=u_n/2$, so that $\sum u_n$ converges. Hence $\sum u_n R_n(x)$ converges a.e., so that $\sum u_n^2$ converges.[4]

(ii) Suppose now that $s=\sum u_n$ and $\sum u_n^2$ converge. Then the series of (1) converges a.e. to a function $\phi(x)$. The $R_n(x)$ are orthonormal, so that by the Riesz-Fischer theorem the series $\sum u_n R_n(x)/2$ converges in the mean to a function of L^2; this function must coincide a.e. with $\phi(x)-s/2$.

To establish (2) note that by the Schwarz inequality

$$\int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right| dx$$

$$\leqq \left(\int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right|^2 dx \right)^{1/2}$$

$$= o(1) \qquad (N \to \infty).$$

Since $\int_0^1 R_n(x)dx=0$, (2) is an immediate consequence.

We conclude with some remarks.

(i) Hill points out that if in our theorem we take $\sum u_n$ to be conditionally convergent then D is of the first category[1] though of measure 1.[5]

(ii) Ulam notes that if $\sum |u_n|$ converges then the set of values taken on by $\phi(x)$ is a perfect set and asks what perfect sets can be obtained this way.[5]

(iii) We can obtain part of the above theorem from the laws of 0 or 1 of probability.[6] For (1) may be regarded as a series of independent random variables of mean-value and standard deviation $u_n/2$.

(iv) The above method furnishes a simple proof of a theorem of Steinhaus: the series (1) is $(C, 1)$ summable a.e. if and only if $\sum u_n$ is $(C, 1)$ summable and $\sum u_n^2$ converges.[7]

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[4] Kacmarcz and Steinhaus, op. cit. p. 234.

[5] Written communication to the author.

[6] See, for example, Khintchine and Kolmogoroff, *Über Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden*, Rec. Math. (Mat. Sbornik) N. S. vol. 32 (1925) p. 668.

[7] Incorporated into a paper of Paley and Zygmund, *On some series of functions* (2), Proc. Cambridge Philos. Soc. vol. 26 (1930) p. 473.

RANDOM SUMMABILITY AND FOURIER SERIES

RICHARD BELLMAN

Fejér's theorem that the first arithmetic mean of the partial sums of the Fourier series of a function converges to the function p.p. is equivalent to saying the average values of the partial sums, taken in order, are close to the function. It is interesting to ask whether, if the partial sums are chosen *at random*, and then averaged, the new averages will be close to the function. We shall show that this is true *most of the time.*

Let us define *random summability* in terms of the existence of the limit

$$\lim_{n\to\infty} \frac{\sum_1^n s_k(x)(1+r_k(t))}{\sum_1^n (1+r_k(t))}$$

where $r_k(t)$ are the Rademacher functions, $r_k(t)=\text{sign sin } 2^{k+1}\pi t$, and $s_k(x)$ are the partial sums of $\sigma(f)=a_0+\sum_1^\infty (a_k \cos kx+b_k \sin kx)$.

The only fact we require concerning the Rademacher functions is [1][1]: If $\sum_1^\infty a_k^2<\infty$, $\sum_1^\infty a_k r_k(t)$ converges p.p. in t.

Now we have

$$\frac{\sum_1^n s_k(x)(1+r_k(t))}{\sum_1^n (1+r_k(t))} = \frac{\sum_1^n s_k(x)}{n}\cdot\frac{n}{\sum_1^n (1+r_k(t))} + \frac{\sum_1^n s_k(x)r_k(t)}{n}\cdot\frac{n}{\sum_1^n (1+r_k(t))}$$

Also $\sum_1^\infty r_k(t)/k^{1/2}\log(k+1)$ converges p.p. in t, by the above, since $\sum_1^\infty 1/k\log(k+1)^2<\infty$, so that $\sum_1^n r_k(t)=o(n^{1/2}\log n)$, by Kronecker's Lemma (much more is known, but this is more than needed). Hence, certainly $\sum_1^n r_k(t)=o(n)$, and $\sum_1^n (1+r_k(t))\sim n$.

By Fejér's theorem $n^{-1}\sum_1^n s_k(x)\to f(x)$ p.p. in x, so that it remains to prove $n^{-1}\sum_1^n s_k(x)r_k(t)\to 0$ p.p. in x and t.

But $\sum_1^\infty (s_k(x)/k)^2<\infty$ p.p. in x, since $s_k(x)=o(\log k)$ p.p. in x [2]. Therefore $\sum_1^\infty s_k(x)r_k(t)/k$ converges p.p. in x and t, and using Kronecker's Lemma again $\sum_1^n s_k(x)r_k(t)=o(n)$.

Thus, we have proved: If $f(x)$ belongs to L, the Fourier series of $f(x)$ is *random summable* p.p. in x, *most of the time.*

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[1] Numbers in brackets indicate references at end of paper.

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1. Zygmund, *Trigonometrical series*, chap. 5, p. 123.
2. Ibid., chap. 2, p. 32.

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ON FIBRE SPACES. II

RALPH H. FOX

This paper is primarily concerned with fibre mappings[1] into an absolute neighborhood retract. Theorem[2] 3 is a converse of the covering homotopy theorem; it characterizes fibre mappings (into a compact ANR) as mappings for which the covering homotopy theorem holds. Theorem 4 is Borsuk's fibre theorem;[3] the proof[4] which I present here is new. It seems to me that this theorem is a promising tool in function-space theory. Also I think that it furnishes conclusive justification for the generality of the Hurewicz-Steenrod definition of a fibre space. In fact, a fibre space of the type constructed by Borsuk's theorem almost never has a compact base space and almost never has its fibres of the same topological type.

The common denominator of the proofs of Theorems 3 and 4 is a property which I call *local equiconnectivity*. Local equiconnectivity is a strengthened form of local contractibility and a weakened form of the absolute neighborhood retract property (Theorems 1 and 2). Definitions and notations are those of FS. I.[5]

Let Δ be the diagonal subset $\sum_{b\in B}(b, b)$ of $B\times B$. I shall call the space B *locally equiconnected* (or, to be specific, (U, V)-equiconnected) if there are neighborhoods U and V of Δ and a homotopy λ in B between the two projections of U which does not move the points of Δ and which is uniform[5] with respect to V. Precisely:

(1) $\lambda_t(b_0, b_1)$ is defined for all $(b_0, b_1)\in U$,

(2) $\lambda_0(b_0, b_1) = b_0$,

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[1] W. Hurewicz and N. Steenrod, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) p. 61.

[2] This theorem was announced in Hurewicz-Steenrod, op. cit. footnote 3.

[3] K. Borsuk, Fund. Math. vol. 28 (1937) p. 99.

[4] This proof was announced in the author's paper *On the deformation retraction of some function spaces* · · · , Ann. of Math. vol. 44 (1943) p. 52.

[5] $\bar{\pi}(x, b) = (\pi(x), b)$ as in R. H. Fox, *On fibre spaces*. I, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 555–557.

(3) $\lambda_1(b_0, b_1) = b_1$,

(4) $\lambda_t(b, b) = b$ for every $(b, b) \in \Delta$, $0 \leqq t \leqq 1$,

(5) there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{(b_0, b_1) \in U} (\lambda_t(b_0, b_1), \lambda_{t'}(b_0, b_1)) \subset V$.

Roughly speaking, B is locally equiconnected if there are paths between sufficiently nearby points such that the paths depend continuously on the end points.

THEOREM 1. *A locally equiconnected space is locally contractible.*

Let N be a neighborhood of some point b_1 of B and let M denote the set of points b_0 such that $\sum_{0 \leqq t \leqq 1} \lambda_t(b_0, b_1) \subset N$. By (4), $b_1 \in M$; a simple continuity argument shows that M is a neighborhood of b_1. Since M is contractible to b_1 in N the theorem is proved.

THEOREM 2. *A compact ANR-set is locally equiconnected.*

Let B be a neighborhood retract of the Hilbert parallelotope Q and let r be a retraction of an open neighborhood N of B onto B. Since $Q - N$ and B are disjoint compact sets $\epsilon = d(B, Q - N)/2 > 0$. Let U_ϵ be the closed neighborhood of Δ determined by the covering of B by ϵ-spheres and let $\lambda_t(b_0, b_1) = r((1-t)b_0 + tb_1)$ for $(b_0, b_1) \in U_\epsilon$, $0 \leqq t \leqq 1$. Conditions (1), (2), (3), and (4) are obviously satisfied. Condition (5) follows, for any V, from the compactness of U_ϵ.

From Theorems 1 and 2 it follows,[6] for finite dimensional compacta, that local contractibility, local equiconnectivity and the ANR property are equivalent. For infinite dimensional spaces no more is known than is implied above.

THEOREM 3 (CONVERSE OF THE COVERING HOMOTOPY THEOREM). *Let B be a (U, V)-equiconnected space and let $\pi \in B^X$. Suppose that for every mapping $g \in X^Y$ and homotopy h in B which is uniform with respect to V and has initial value[5] πg there exists a covering homotopy h^* in X with initial value g. Then π is a fibre mapping relative to U.*

Let $h_t(x, b) = \lambda_t(\pi(x), b)$. Since h is uniform with respect to V there is a covering homotopy h^* such that $h_0^*(x, b) = x$. Let $\phi(x, b) = h_1^*(x, b)$. Then[5] ϕ maps $\pi^{-1}(U)$ continuously into X and $\pi\phi(x, b) = b$. Since $h_{[0,1]}(x, \pi(x)) = \pi(x)$ it follows that $\phi(x, \pi(x)) = h_1^*(x, \pi(x)) = h_0^*(x, \pi(x)) = x$. Thus ϕ is a slicing function.

Let A be a closed subset of X and let π denote the sectioning operation $\pi(f) = f|A$, $f \in Y^X$.

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[6] K. Borsuk, Fund. Math. vol. 19 (1932) p. 240, Theorem 32.

THEOREM 4 (BORSUK'S FIBRE THEOREM). *If A is closed in X and Y is a compact ANR-set then π is a fibre mapping.*

By Theorem 2, Y is locally equiconnected and, if it is suitably metrized, there is a positive number ϵ such that $\lambda_t(y_0, y_1)$ is defined whenever $d(y_0, y_1) < \epsilon$. Let Γ_0 denote the graph of π and let Γ_ϵ denote the subset of $Y^X \times Y^A$ defined by the rule $(f, g) \in \Gamma_\epsilon$ when $d(\pi(f), g) < \epsilon$. Because Y is compact Γ_ϵ is a neighborhood of Γ_0. Define

$$\psi_t(f, g, x) = \begin{cases} \lambda_t(f(x), g(x)) & \text{for } (f, g, x) \in \Gamma_\epsilon \times A, \\ f(x) & \text{for } (f, g, x) \in \Gamma_0 \times X. \end{cases}$$

Thus ψ is a homotopy in Y; each ψ_t is defined on the closed subset $C = \Gamma_\epsilon \times A + \Gamma_0 \times X$ of $\Gamma_\epsilon \times A$. But $\psi_0(f, g, x) = f(x)$ for every $(f, g, x) \in C$, and this map has the extension $\psi_0^*(f, g, x) = f(x)$ defined for every $(f, g, x) \in \Gamma_\epsilon \times X$. It follows[7] that ψ_1 can be extended to $\Gamma_\epsilon \times X$. Let ψ_1^* denote an extension of ψ_1 and set $\phi(f, g)(x) = \psi_1^*(f, g, x)$ for $(f, g) \in \Gamma_\epsilon$ and $x \in X$, so that $\phi(f, g) \in Y^X$ for every fixed $(f, g) \in \Gamma_\epsilon$. Then ϕ maps Γ_ϵ into Y^X, $\pi\phi(f, g) = g$, $\phi(f, \pi(f)) = f$. Thus ϕ is a slicing function for π.

Since the image set of a fibre mapping is necessarily open and closed in the base space, an example[8] "$\mathcal{E}$" shows that Theorem 4 is false for non-compact ANR-sets Y. However if neither X nor Y are compact (as in "$\mathcal{E}$") the topology of Y^X (and also of Y^A) depends on the metrization of Y. Thus it may be possible (as it is in "$\mathcal{E}$") to remetrize an ANR-set Y so as to make the sectioning operations fibre mappings. It should be observed that Borsuk has shown that Theorem 4 is false (with or without remetrization) if Y is not locally contractible.[4]

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[7] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 86.

[8] R. H. Fox, Bull. Amer. Math. Soc. vol. 48 (1942) p. 271 footnote 3.

EXTENSIONS OF CERTAIN DYNAMICAL THEOREMS OF HALPHEN AND KASNER

JOHN DeCICCO

1. Introduction. We wish to present here some extensions of certain theorems of Halphen and Kasner concerning the dynamical trajectories of positional fields of force in space.

Kasner has developed the differential geometry of the dynamical trajectories of general positional fields of force in the plane and in space in his Princeton Colloquium Lectures.[1] Recently Kasner and the author have introduced the concept of *generalized* fields of force which depend not only upon the position of the point but also upon the direction through the point.[2] In a generalized field of force in space, there are ∞^5 dynamical trajectories just as in the ordinary positional case.

The theorem of Halphen which we wish to extend to generalized fields of force is that all positional fields of force whose ∞^5 dynamical trajectories are plane curves are of the central or parallel type. That is, the lines of force are all straight lines through a fixed point O (which may be finite or at infinity). Moreover the ∞^5 trajectories consist of the ∞^2 systems of ∞^3 ordinary plane dynamical trajectories of the central or parallel type, each such system lying in a plane through the point O. There are $\infty^{3+f(3)}$ such systems of ∞^5 dynamical trajectories,[3] and the number of central fields of force is $\infty^{4+f(3)}$.

We find that all generalized fields of force whose ∞^5 dynamical trajectories are plane curves form a more extensive class. The ∞^5 trajectories consist, in general, of ∞^2 systems of ∞^3 generalized plane dynamical trajectories, each such system lying in a plane tangent to a base surface Σ. The various degenerate situations are four in number and may be described as follows: (i) The base planes are tangent to a given curve C, (ii) the base planes pass through a given finite point O, (iii) the base planes are all parallel to the tangent planes of a given

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[1] Kasner, *Differential geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934. Also see Trans. Amer. Math. Soc. vols. 7–11 (1906–1910).

[2] See (1) *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 23–38; and (2) *Generalized dynamical trajectories in space* abstract 49-3-120.

[3] The symbol $\infty^{k f(n)}$ denotes the content of a geometric manifold which depends on k functions of n variables only. See Kasner, *A notation for infinite manifolds*, Amer. Math. Monthly vol. 49 (1942) pp. 243–244.

cone Γ, and (iv) the base planes are all parallel to a fixed line L. The number of such planar systems of generalized dynamical trajectories is $\infty^{f(2)+2f(5)}$, and the number of associated generalized fields of force is $\infty^{f(2)+f(4)+2f(5)}$.

The theorem of Kasner which we wish to generalize is concerned with arbitrary positional fields of force and may be stated in the following manner. If a particle starts at rest from any given point O the resulting rest trajectory which it describes will be initially tangent to, and also will possess the same osculating plane as, the line of force at O; moreover, the ratio of the curvature of the rest trajectory to that of the line of force is 1:3.

In a generalized field of force this theorem is no longer valid. It is true that the rest trajectory and the line of force will be initially tangent, but, in general, they will have different osculating planes at O. We determine all those fields of force such that, at any point O of space, the rest trajectory and the line of force will have the same osculating plane. In the final part of our paper, we obtain a converse of Kasner's theorem.

2. The differential equations of the ∞^5 generalized dynamical trajectories. We now consider any generalized field of force in space. Observe that an ordinary positional field of force could be called *isotropic* whereas a generalized field of force could be termed *anisotropic*. (A state of stress in a solid medium is an example of a generalized field of force, although it is not of the most general character.)

Let $F(\phi, \psi, \chi)$ be the rectangular components of the generalized force vector which corresponds to the lineal element $E(x, y, z, y', z')$. (Note that the primes denote differentiation with respect to x). We assume that the direction of our force vector $F(\phi, \psi, \chi)$ does not identically agree with that of the corresponding lineal element E. The equations of motion of a particle of unit mass are

$$(1) \qquad \begin{aligned} d^2x/dt^2 &= \phi(x, y, z, y', z'), \qquad d^2y/dt^2 = \psi(x, y, z, y', z'), \\ d^2z/dt^2 &= \chi(x, y, z, y', z'), \end{aligned}$$

where, of course, t is the time.

Upon eliminating the time t from these equations, it is found that the ∞^5 dynamical trajectories of a generalized field of force are given as the integral curves of either of the two Monge differential equations of third order

$$(2.1) \qquad \begin{aligned} (\psi - y'\phi)y''' = {} & [(\psi_x + y'\psi_y + z'\psi_z) - y'(\phi_x + y'\phi_y + z'\phi_z)]y'' \\ & + [\psi_{y'} - y'\phi_{y'} - 3\phi]y''^2 + (\psi_{z'} - y'\phi_{z'})y''z'', \end{aligned}$$

or

$$(2.2)\qquad (\chi - z'\phi)z''' = [(\chi_x + y'\chi_y + z'\chi_z) - z'(\phi_x + y'\phi_y + z'\phi_z)]z'' + [\chi_{z'} - z'\phi_{z'} - 3\phi]z''^2 + (\chi_{y'} - z'\phi_{y'})y''z'',$$

solved together with the Monge differential equation of second order

$$(3)\qquad z'' = [(\chi - z'\phi)/(\psi - y'\phi)]y''.$$

Therefore the generalized dynamical trajectories are of *the three-dimensional type* (G), namely

$$(G)\qquad \begin{aligned} z'' &= K(x, y, z, y', z')y'', \\ y''' &= G(x, y, z, y', z')y'' + H(x, y, z, y', z')y''^2. \end{aligned}$$

Conversely any system of ∞^5 curves of the three-dimensional type (G) represents the dynamical trajectories of $\infty^{f(4)}$ generalized fields of force.

Kasner has shown that the three-dimensional type (G) may be characterized by the two geometric properties listed below. For each of the ∞^1 trajectories passing through a given lineal element E, construct the osculating plane and the osculating sphere at E. The two properties are:

Property I. The ∞^1 trajectories possess the same osculating plane at E.

Property II. The locus of the centers of the osculating spheres at E is a straight line.

3. **Extension of Halphen's theorem.** Now we shall discuss our extension of the Halphen problem.

THEOREM 1. *A system of ∞^5 generalized dynamical trajectories consists wholly of plane curves if and only if it is represented by a system of differential equations of the three-dimensional type* (G) *where the function K is defined as a solution of an equation of the form*

$$(4)\qquad F[z' - y'K, K, z - x(z' - y'K) - yK] = 0,$$

where F is an arbitrary function of three variables.

There are thus $\infty^{f(2)+2f(5)}$ systems of generalized dynamical trajectories which consist wholly of plane curves.

We proceed with the proof of Theorem 1. Upon substituting the three-dimensional type (G) into the condition $y''z''' - y'''z'' = 0$ for plane curves and simplifying, we obtain the equation

$$(5)\qquad y''^3(K_{y'} + KK_{z'}) + y''^2(K_x + y'K_y + z'K_z) = 0.$$

By this equation we see that, upon excluding the ∞^4 straight lines which are trajectories of any three-dimensional type (G), there are *at most* ∞^4 trajectories in any system of ∞^5 dynamical trajectories, which are plane curves.

Now if our three-dimensional type (G) is to contain more than ∞^4 plane trajectories, we observe that (5) must be an identity in y''. Therefore the function K must satisfy the two partial differential equations of the first order

$$K_{y'} + KK_{z'} = 0, \qquad K_x + y'K_y + z'K_z = 0. \tag{6}$$

First let us suppose that K is independent of y' or z'. From these equations we see that K must be constant. This solution for K is given by the equation (4) where F is a function of the second variable K alone. Therefore, we can henceforth suppose that K contains y' or z' explicitly.

By the first of equations (6) and by the fact that K contains y' or z', we find that K must be a solution of a finite equation of the form

$$z' - y'K = \lambda(K, x, y, z), \tag{7}$$

where λ is an arbitrary function of four variables.[4] Obtaining the partial derivatives of K with respect to x, y, and z from this equation and substituting into the second equation of (6), we find

$$\lambda_x + y'\lambda_y + z'\lambda_z = 0. \tag{8}$$

Eliminating z' from this equation and equation (7), we find

$$y'(\lambda_y + K\lambda_z) + (\lambda_x + \lambda\lambda_z) = 0. \tag{9}$$

Now this must be an identity in y'. Assume the contrary. We then can solve this equation for y' and differentiate the result with respect to z'. Since, by (7), $K_{z'} \neq 0$ we find

$$\frac{\partial}{\partial K}\left(\frac{\lambda_x + \lambda\lambda_z}{\lambda_y + K\lambda_z}\right) = 0. \tag{10}$$

This will mean that the equation (9) solved for y' is independent of y'. This is absurd. Hence (9) is an identity in y'.

By the preceding argument, the function λ of (K, x, y, z) must satisfy the two partial differential equations of the first order

$$\lambda_y + K\lambda_z = 0, \qquad \lambda_x + \lambda\lambda_z = 0. \tag{11}$$

[4] This equation (7) arises in the characterization of ordinary positional fields of force in space. See Kasner, *Dynamical trajectories: The motion of a particle in an arbitrary field of force*, Trans. Amer. Math. Soc. vol. 8 (1907) pp. 135–158.

Suppose now that $\lambda_z = 0$. These equations show that λ is independent of (x, y, z). Therefore λ is a function of K only. Substituting this result into (7), we discover that the function K is given by the equation (4) where F is independent of the third variable. Henceforth we may suppose that $\lambda_z \neq 0$.

Since $\lambda_z \neq 0$, we find from the second of equations (11) that λ must satisfy an equation of the form

$$z - x\lambda = \mu(\lambda, K, y), \tag{12}$$

where μ is an arbitrary function of three variables. Obtaining the partial derivatives of λ with respect to y and z from this equation, and substituting into the first of equations (11), we find

$$\mu_y = K. \tag{13}$$

The integration of this equation yields the following solution for μ,

$$\mu = yK + f(\lambda, K), \tag{14}$$

where f is an arbitrary function of two variables. Substituting this into (12), we find

$$z = x\lambda + yK + f(\lambda, K). \tag{15}$$

Finally, eliminating λ from this equation and equation (7), we discover that

$$z = x(z' - y'K) + yK + f(z' - y'K, K). \tag{16}$$

This may be obtained as a solution of the equation (4) for the third variable. Thus the proof of Theorem 1 is complete.

To obtain Halphen's theorem as a consequence of Theorem 1, we first observe that, for the ordinary positional case, equation (3) demonstrates that equation (4) must be a function of a linear integral function in $(z' - Ky')$ and K with coefficients functions of (x, y, z) only. If the third variable is missing in (4), then the coefficients must be constants and the field of force is of the parallel type. On the other hand, if the third variable is present in (4), the equation may be written in the form (16), where f is integral linear in the two variables with constant coefficients. Solving the resulting equation for K, we find that the field of force is of the central type. This completes the proof of Halphen's theorem.

THEOREM 2. *The ∞^5 generalized dynamical trajectories of Theorem 1 consist of ∞^2 systems of ∞^3 generalized plane dynamical trajectories, each such system lying in a plane which is, in general, tangent to a base surface Σ.*

The degenerate situations which arise are four in number and may be described as follows: (i) The base planes are tangent to a given curve C, (ii) the base planes pass through a given finite point O, (iii) the base planes are all parallel to the tangent planes of a given cone Γ, and (iv) the base planes are all parallel to a fixed line L.

To prove Theorem 2, we first observe that the differential equations of the dynamical trajectories of Theorem 1 may be written as

$$(17)\qquad \begin{aligned} &F[z' - y'z''/y'', z''/y'', z - x(z' - y'z''/y'') - yz''/y''] = 0,\\ &y''' = Gy'' + Hy''^2. \end{aligned}$$

To find out what curves of this system (17) lie in the plane $z=ax+by+c$, we eliminate z between the first of equations (17) and the equation of the plane. The result of this elimination yields

$$(18)\qquad F(a, b, c) = 0.$$

Therefore we conclude that for a plane to contain at least one curve of (17), its parameters (a, b, c) must satisfy the preceding equation. Moreover any such plane will contain ∞^3 spatial generalized dynamical trajectories, and these form a generalized plane dynamical system of arbitrary character. The conclusion of our theorem is obtained from the observation that the ∞^2 planes whose parameters (a, b, c) satisfy (18) are, in general, tangent to a surface Σ. The degenerate situations also can be obtained by a discussion of the various forms that the function F may assume.

It must be noted here that any plane system of ∞^3 generalized dynamical trajectories of Theorem 2 is of perfectly general character, whereas in Halphen's theorem any such plane system must be of the central or parallel type.

4. The lines of force and the rest trajectories. In the remainder of this paper, we shall suppose that

$$(19.1)\qquad (\psi_{y'} - y'\phi_{y'} - \phi)(\chi_{z'} - z'\phi_{z'} - \phi) - (\psi_{z'} - y'\phi_{z'})(\chi_{y'} - z'\phi_{y'}) \neq 0$$

and

$$(19.2)\qquad (\psi_{y'} - y'\phi_{y'} - 3\phi)(\chi_{z'} - z'\phi_{z'} - 3\phi) - (\psi_{z'} - y'\phi_{z'})(\chi_{y'} - z'\phi_{y'}) \neq 0.$$

Let us first of all consider the lines of force in a generalized field of force. At any point O there is, in general, one (and at most ∞^1) lineal element E_0 whose direction coincides with that of the corresponding force vector F_0. The lines of force are the integral curves of the differential equations defined by all the lineal elements E_0 of a certain

region of space. This system of differential equations is

$$\psi - y'\phi = 0, \qquad \chi - z'\phi = 0. \tag{20}$$

By (19.1), it follows that there are, in general, ∞^2 lines of force. Otherwise if the left-hand side of (19.1) is identically zero, there are $\infty^{f(1)}$, or ∞^1, or a finite number of lines of force.

Consider next the rest trajectories in a generalized field of force. A rest trajectory is defined as the trajectory described by a particle of unit mass which starts at rest from any point O. By (19.2) there are, in general, ∞^3 rest trajectories in a generalized field of force. Otherwise if the left-hand member of (19.2) is identically zero, there are at most $\infty^{f(1)}$ rest trajectories.

Thus it is obvious that under the assumptions (19.1) and (19.2) there are, in general, ∞^2 lines of force and ∞^3 rest trajectories.

5. **The extension of Kasner's theorem concerning one-third of the curvatures.** In this section we shall determine all generalized fields of force such that, at any point O, the rest trajectory and the line of force will have the same osculating plane.

In the first place, we can easily show that the rest trajectory through the point O is tangent to the line of force at O. This is an immediate consequence of the equations

$$\begin{aligned} dy/dt &= y'dx/dt, \qquad d^2y/dt^2 = y'd^2x/dt^2 + y''(dx/dt)^2, \\ dz/dt &= z'dx/dt, \qquad d^2z/dt^2 = z'd^2x/dt^2 + z''(dx/dt)^2, \end{aligned} \tag{21}$$

together with the initial conditions $(dx/dt)_0 = (dy/dt)_0 = (dz/dt)_0 = 0$ and the equations (20) defining the lines of force.

We discuss our extension of Kasner's theorem.

THEOREM 3. *The rectangular components $F(\phi, \psi, \chi)$ of any generalized field of force for which, at any point O of space, the rest trajectory and the line of force possess the same osculating plane at O must be such that the single eliminant with respect to y' and z' of the three equations*

$$\psi - y'\phi = 0, \quad \chi - z'\phi = 0, \quad (AC + BF)/(DC + EF) = C/F, \tag{22}$$

where A, B, C, D, E, F are defined by the expressions

$$\begin{aligned} A &= \psi_{y'} - y'\phi_{y'}, & B &= \psi_{z'} - y'\phi_{z'}, \\ C &= (\psi_x + y'\psi_y + z'\psi_z) - y'(\phi_x + y'\phi_y + z'\phi_z), \\ D &= \chi_{y'} - z'\phi_{y'}, & E &= \chi_{z'} - z'\phi_{z'}, \\ F &= (\chi_x + y'\chi_y + z'\chi_z) - z'(\phi_x + y'\phi_y + z'\phi_z), \end{aligned} \tag{23}$$

is identically zero.

In the first place, the first derivatives (y', z') of the line of force and the rest trajectory through the point O are given as solutions of the equations (20).

By obtaining the total derivatives of equations (20) with respect to x and using the formulas (23), we find that the equations defining the second derivatives (Y'', Z'') of the line of force through the point O are

$$(24) \quad (A - \phi)Y'' + BZ'' + C = 0, \quad DY'' + (E - \phi)Z'' + F = 0.$$

By equations (2.1) and (2.2), we see that the equations defining the second derivatives (y'', z'') of the rest trajectory through the point O are

$$(25) \quad (A - 3\phi)y'' + Bz'' + C = 0, \quad Dy'' + (E - 3\phi)z'' + F = 0.$$

It is obvious that, in general, the osculating planes can not be identical. Let us impose the condition that the two osculating planes be identical. Then there must exist a number ρ such that

$$(26) \quad y'' = \rho Y'', \quad z'' = \rho Z''.$$

Eliminating Y'' and Z'' from equations (24), (25), and (26), we find

$$(27) \quad \begin{aligned} y'' &= (1 - \rho)C/2\phi, \quad z'' = (1 - \rho)F/2\phi, \\ \rho &= (AC + BF - \phi C)/(AC + BF - 3\phi C) \\ &= (DC + EF - \phi F)/(DC + EF - 3\phi F). \end{aligned}$$

This last equation establishes the validity of Theorem 3.

For the generalized fields of force of Theorem 3, let λ be defined by the two equal expressions

$$(28) \quad \lambda = (AC + BF)/\phi C = (DC + EF)/\phi F.$$

THEOREM 4. *For the generalized fields of force of Theorem* 3, *the ratio* ρ *of the curvature of the rest trajectory to that of the line of force is given by the formula*

$$(29) \quad \rho = (1 - \lambda)/(3 - \lambda).$$

This result may be considered to be an extension of Kasner's one-third theorem.

6. **A converse of Kasner's theorem concerning one-third the curvatures.** If, in addition to the possession of the same osculating plane, we impose the condition that the ratio $\rho = 1/3$, we obtain the following converse of Kasner's theorem.

THEOREM 5. *The rectangular components $F(\phi, \psi, \chi)$ of any generalized field of force for which, at any point O of space, the rest trajectory and the line of force possess the same osculating plane and the ratio ρ of the curvature of the rest trajectory to that of the line of force is $1/3$ must be such that the two eliminants with respect to y' and z' of the four equations*

$$\begin{aligned} \psi - y'\phi &= 0, & \chi - z'\phi &= 0, \\ AC + BF &= 0, & DC + EF &= 0, \end{aligned} \tag{30}$$

where (A, B, C, D, E, F) are defined by equations (23), are identically zero.

Thus it is seen that, in the generalized fields of force in space, there are no general geometric relations between the lines of force and the rest trajectories except for the trivial one concerning the possession of the common tangent line. This is entirely different from the two-dimensional case (see footnote 2) where there is a theory—the general theorem being quite similar to our Theorem 4.

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UNIFORM CONVEXITY. III

MAHLON M. DAY

It is the purpose of this note to fill out certain results given in two recent papers on uniform convexity of normed vector spaces.[1] A normed vector space[2] B is called *uniformly convex* with *modulus of convexity* δ if for each $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that for every two points b and b' of B satisfying the conditions $\|b\|=\|b'\|=1$ and $\|b-b'\|\geqq\epsilon$ the quantity $\|b+b'\|\leqq 2(1-\delta(\epsilon))$. If $\|b_0\|=1$, B is said to be *locally uniformly convex near* b_0 if there is a sphere about b_0 in which the condition for uniform convexity holds. Theorem 1 shows that all properties of normed vector spaces which are invariant under isomorphism are the same for uniformly convex and locally uniformly convex spaces. Theorem 2 gives a necessary condition for isomorphism with a uniformly convex space. The condition is in terms of isomorphisms of finite dimensional subspaces and is suggested by examples given in [I]; it is not known whether the condition is sufficient. Theorem 3 is somewhat more general than Theorem 3 of [II]; it uses uniformly convex function spaces instead of the l_p spaces of [II].

A *cone* C in B is a set which contains all of every half line from the origin through each point of C.

LEMMA 1. *A normed vector space B is locally uniformly convex near b_0 if and only if there exists a convex cone C, with b_0 in its interior, such that for every ϵ there is a $\delta_1(\epsilon)>0$ such that the conditions $\|b\|\leqq 1$, $\|b'\|\leqq 1$, and $\|b-b'\|\geqq\epsilon$ imply $\|b+b'\|\leqq 2(1-\delta_1(\epsilon))$ for every pair of points b and b' in C.*

If this condition is satisfied there is obviously a sphere about b_0 inside C, so that in that sphere $\delta(\epsilon)$ can be taken equal to $\delta_1(\epsilon)$. On the other hand, if there is a sphere of radius $2k$ about b_0 in which δ can be defined, it can be shown that it suffices to let C be the cone through points of the sphere of radius k about b_0 and to let $\delta_1(\epsilon)=\inf\,[\epsilon/10,\ \delta(4\epsilon/5)/2]$.

LEMMA 2. *If the cone C of Lemma 1 contains a sphere about b_0 of radius k, if $\|b\|\leqq 1$ and if $\|b-b_0\|\geqq k$, then $\|b+b_0\|<2-\delta_1(k)$.*

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[1] These papers are [I] *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317, and [II] *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 504–507.

[2] See Banach, *Théorie des opérations linéaires*, Warsaw, 1932, for general definitions.

This is obvious if $\|b-b_0\|=k$. If $\|b\|\leqq 1$ and $\|b-b_0\|>k$ there exists a point $b_1=\lambda b+(1-\lambda)b_0$, $0<\lambda<1$, on the line segment from b_0 to b such that $\|b_0-b_1\|=k$ while $\|b_1\|\leqq 1$; hence $\|b_1+b_0\|\leqq 2(1-\delta_1(k))$. Let f be a linear functional such that $f(b')\leqq\|b'\|$ for all b' in B and such that the line $\{b'|f(b')=1\}$ in the plane of 0, b_0 and b touches the unit sphere in B at the point of intersection of that sphere with the half line from 0 through $b+b_0$, so that $f(b+b_0)=\|b+b_0\|$. Then $f(b_0+b_1)\leqq\|b_0+b_1\|\leqq 2(1-\delta_1(k))$. Two cases can now be distinguished: If $f(b_0)\geqq f(b)$, $2-2\delta_1(k)\geqq f(b_1+b_0)\geqq f(b+b_0)=\|b+b_0\|$. If $f(b_0)<f(b)$, $2(1-\delta_1(k))\geqq f(b_1+b_0)=f(b_1)+f(b_0)>2f(b_0)$, so $\|b+b_0\|=f(b+b_0)=f(b_0)+f(b)<1-\delta_1(k)+1$.

THEOREM 1. *If B is locally uniformly convex near some point b_0, then B is isomorphic to a uniformly convex space. If k is the radius of the sphere which exists by Lemma 1 about b_0, a suitable modulus of convexity for the new space is given in terms of the old by* $\delta_1'(\epsilon)=1-1/[1+\delta_1(\delta_1(k)\epsilon/4)/(k+\delta_1(k)/4)]$.

Suppose the cone C of Lemma 1 contains a sphere $\{b|\|b-b_0\|\leqq k\}$ about b_0; let $\alpha=1-\delta_1(k)/4$ and consider the two spheres $E_1=\{b|\|b-\alpha b_0\|\leqq 1\}$ and $E_2=\{b|\|b+\alpha b_0\|\leqq 1\}$. If S is the intersection of E_1 and E_2, it is clear that S is convex and symmetric about the origin, and that S contains the sphere $\{b|\|b\|\leqq\delta_1(k)/4\}$.

To show $\|b\|\leqq k+\delta_1(k)/4$ for each b in S, it suffices to show that $b\in S$ implies that $b+\alpha b_0$ is within k of b_0. If this is false, that is, if $\|b+\alpha b_0-b_0\|>k$, then, by Lemma 2, $\|b+\alpha b_0+b_0\|<2-\delta_1(k)$. However $\|b+\alpha b_0+b_0\|=\|b-\alpha b_0+(1+2\alpha)b_0\|\geqq(1+2\alpha)\|b_0\|-\|b-\alpha b_0\|\geqq 1+2\alpha-1=2\alpha=2-\delta_1(k)/2>2-\delta_1(k)$.

Let $|b|$ be the smallest non-negative value of t for which the point b/t is in S. Then $|\cdots|$ defines a new norm in B and it is clear from the inequalities thus far derived that $[\delta_1(k)/4]|b|\leqq\|b\|\leqq[k+\delta_1(k)/4]|b|$, so this new norm defines a space isomorphic to the original and all that need be proved is that $|\cdots|$ is uniformly convex. If b_1, $b_2\in S$, and $|b_1-b_2|>\epsilon$, then $\|(b_1+\alpha b_0)-(b_2+\alpha b_0)\|=\|b_1-b_2\|\geqq\delta_1(k)\epsilon/4$. Also $\|b_i+\alpha b_0-b_0\|\leqq k$ by the preceding paragraph so, by the original hypotheses near b_0, $\|b_1+b_2+2\alpha b_0\|\leqq 2[1-\delta_1(\delta_1(k)\epsilon/4)]\equiv 2(1-\mu(\epsilon))$; that is, $(b_1+b_2)/2=E_1'=\{b|\|b+\alpha b_0\|\leqq 1-\mu(\epsilon)\}$. The same argument with $-\alpha b_0$ and $-b_0$ shows that $(b_1+b_2)/2\in E_2'=\{b|\|b-\alpha b_0\|\leqq 1-\mu(\epsilon)\}$.

It will now suffice to show that there is a $\delta_1'(\epsilon)>0$ such that $|b|<1-\delta_1'(\epsilon)$ if $b\in E_1'\cdot E_2'$. $E_i'\subset E_i$, $i=1, 2$, so for any b in $E_1'\cdot E_2'$ there is a number $t\geqq 1$ such that $|tb|=1$; hence, either $\|tb+\alpha b_0\|=1$ or $\|tb-\alpha b_0\|=1$. These cases are interchanged by replacing b by

$-b$ so it suffices to consider the first; then $1-\mu(\epsilon)\geqq\|ab_0-b\| = \|ab_0-tb+tb-b\| \geqq \|ab_0-tb\|-\|tb-b\| = 1-(t-1)\|b\|$. Therefore $(t-1)\|b\|\geqq\mu(\epsilon)$ or $t\geqq 1+\mu(\epsilon)/\|b\|\geqq 1+\delta_1(\delta_1(k)\epsilon/4)/[k+\delta_1(k)/4]$. Letting $1-\delta_1'(\epsilon)$ be the reciprocal of the last term in the preceding inequality gives $|b| = 1/t\leqq 1-\delta_1'(\epsilon)$ if $b\in E_1'\cdot E_2'$.

We turn now to a necessary condition for isomorphism of B with a uniformly convex space. The effect of uniform convexity on the finite dimensional subspaces of an isomorphic space was used implicitly in [I]; it is given explicit formulation here. Let B_0 and B be two normed vector spaces; then there exist linear operations of norm $\leqq 1$ defined on B_0 with values in B. For each such operator U there is a largest number k_U, $0\leqq k_U\leqq 1$, such that $\|b_0\|\geqq\|U(b_0)\| \geqq k_U\|b_0\|$ for each b_0 in B_0, and this number k_U can be taken as a measure of the distortion of B_0 under the mapping U into B. Define $k(B_0, B)$ to be the least upper bound of k_U as U runs over the linear operators from B_0 to B of norm not greater than 1; explicitly, $k(B_0, B)=\sup_{\|U\|\leqq 1}\inf_{\|b_0\|=1}\|U(b_0)\|$. $k(B_0, B)$ is then a measure of how nearly B_0 approaches isometry with a subspace of B; if $k(B_0, B)=1$, there are operations which come arbitrarily near preserving distances; $k(B_0, B)>0$ if and only if B_0 is isomorphic to a subspace of B. For the present it suffices to choose certain finite dimensional spaces for B_0. Let M_n and L_n be the n-dimensional spaces of sequences $t=(t_1, \cdots, t_n)$ of n real numbers, where $\|t\|_{M_n}=\|(t_1, \cdots, t_n)\|_{M_n}=\max_{1\leqq i\leqq n}|t_i|$ and $\|t\|_{L_n}=\|(t_1, \cdots, t_n)\|_{L_n} =\sum_{1\leqq i\leqq n}|t_i|$. Then $k(M_n, B)=k(L_n, B)=0$ if and only if the dimension of B is less than n; also $k(M_n, B)$ and $k(L_n, B)$ are nonincreasing functions of n for each B.

LEMMA 3. *If U is a one-to-one linear operator from B_1 onto B_2 such that for some $a\geqq 0$, $\|b_1\|\geqq\|Ub_1\|\geqq a\|b_1\|$ for each b_1 in B_1, then for any normed vector space T, $k(T, B_1)\geqq ak(T, B_2)\geqq a^2k(T, B_1)$.*

If $a=0$, this is obvious. If $a>0$ and F is any linear operator from T into B, with $\|F\|\leqq 1$, let UF be defined by $UF(t)=U(F(t))$ for every t in T. Then $\|UF\|\leqq 1$ and $\|UF(t)\|\geqq a\|F(t)\|$ for every t. Hence $\inf_{\|t\|=1}\|UF(t)\|\geqq a\inf_{\|t\|=1}\|F(t)\|$ so $k(T, B_2)\geqq ak(T, B_1)$. If F' is any linear operator of norm $\leqq 1$ from T into B_2, the same argument, using the operator $aU^{-1}F'$, shows that $k(T, B_1)\geqq ak(T, B_2)$.

Note that if U maps B_1 on only part of B_2 or is not 1-1 but is of norm $\leqq 1$, the first half of the proof still holds (although $a=0$ in the second case); it follows that if B_1 is a subspace of B_2, then $k(T, B_1)\leqq k(T, B_2)$.

Sobczyk[3] has defined a special embedding of l_1 into m which can easily be modified to define an isometry of L_{n+1} and a subspace of M_{2^n} so $k(L_{n+1}, B) \geqq k(M_{2^n}, B)$ for every integer n. In particular, L_2 and M_2 are isometric so $k(L_2, B) = k(M_2, B)$.

LEMMA 4. *If δ_1 satisfies Lemma 1 in the whole unit sphere of B and is continuous on the left, then*

(1) $k(M_n, B) \leqq [1 - \delta_1(2k(M_n, B))]^{n-1}$,

(2) $k(L_{2^n}, B) \leqq [1 - \delta_1(2k(L_{2^n}, B))]^n$.

If F is an operation from M_n into B such that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$ for all t, where $k > 0$, let $\epsilon_i = \pm 1$ for $i = 1, \cdots, n$; then the points $F(\epsilon_1, \cdots, \epsilon_n)$ lie in the unit sphere of B since $\|F(\epsilon_1, \cdots, \epsilon_n)\| \leqq \|\epsilon_1, \cdots, \epsilon_n\| = 1$. If $\epsilon_1, \cdots, \epsilon_n$ and $\epsilon_1', \cdots, \epsilon_n'$ are different, $\|F(\epsilon_1, \cdots, \epsilon_n) - F(\epsilon_1', \cdots, \epsilon_n')\| \geqq k\|(\epsilon_1, \cdots, \epsilon_n) - (\epsilon_1', \cdots, \epsilon_n)\| = 2k$; hence $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)\| = \|F(\epsilon_1, \cdots, \epsilon_{n-1}, 1) - F(\epsilon_1, \cdots, \epsilon_{n-1}, -1)\|/2 \leqq 1 - \delta_1(2k)$; that is, $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)/[1 - \delta_1(2k)]\| \leqq 1$ for all $\epsilon_1, \cdots, \epsilon_{n-1}$. These points are at least $2k$ apart for different ϵ_i, so this process can be applied $n-1$ times to show that $\|F(1, 0, 0, \cdots, 0)/[1 - \delta_1(2k)]^{n-1}\| \leqq 1$. Hence $k = k\|1, 0, 0, \cdots, 0\| \leqq \|F(1, 0, 0, \cdots, 0)\| \leqq [1 - \delta_1(2k)]^{n-1}$. Taking $k = k(M_n, B)$ or, if that is impossible, taking the limit as k increases toward $k(M_n, B)$ gives (1).

If F maps L_{2^n} into B so that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$, $k > 0$, for all t, the same sort of argument can be carried through using the points of L_{2^n} which have one coordinate equal to one, the others all zero. It leads to the inequality $k = k\|(2^{-n}, \cdots, 2^{-n})\| \leqq \|F(2^{-n}, \cdots, 2^{-n})\| \leqq [1 - \delta_1(2k)]^n$ which gives (2).

THEOREM 2. *If B is isomorphic to a space which is locally uniformly convex near any point, then* $\lim_n k(M_n, B) = \lim_n k(L_n, B) = 0$.

By Theorem 1, B is isomorphic to a uniformly convex space B'. By Lemma 4, $k(L_{2^n}, B') < [1 - \delta_1(2k(L_{2^n}, B'))]^n$ for all n. If $k(L_{2^n}, B') > k > 0$ for all n, then $0 < k \leqq k(L_{2^n}, B') \leqq (1 - \delta_1(2k))^n \to 0$ as $n \to \infty$; this contradiction and the monotony of $k(L_n, B')$ show that $k(L_n, B') \to 0$. Lemma 3 shows that $k(L_n, B) \to 0$ also. A similar proof holds for $k(M_n, B)$; this can also be proved by using the remark before Lemma 4 and the fact that $k(L_n, B) \to 0$.

This theorem has as a corollary the result of [1]: *If $B = \mathcal{P}^p(B_i)$, where $B_i = l^{p_i}$ or L^{p_i}, and if the numbers p_i are not bounded away from*

[3] A. Sobczyk, *Projection of the space m on its subspace c_0*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 938–947; the construction is given in the proof of Theorem 3.

1 and ∞, then B is not isomorphic to a uniformly convex space.

It is not difficult to give a direct proof of Theorem 2 not using Theorem 1. I have also shown that if B^* is uniformly convex, then $k(L_n, B) \to 0$ (as does $k(M_n, B)$); whether this condition is sufficient as well as necessary for isomorphism of B or B^* with a uniformly convex space is a question which I am, so far, unable to answer.

Some remarks may be made about the minimum values, $k(L_n)$ and $k(M_n)$, of $k(L_n, B)$ and $k(M_n, B)$ taken for n fixed and B varying over the spaces of dimension at least n. $k(M_n, l^p) = n^{-1/p}$ if $2 \leqq p < \infty$ and $k(L_n, l^p) = n^{-1/p'}$ if $1 \leqq p \leqq 2$ where $1/p' + 1/p = 1$. Hence $k(L_n) \leqq k(L_n, l^2) = n^{-1/2}$ and $k(M_n) \leqq k(M_n, l^2) = n^{-1/2}$ for all n. The plane with a regular hexagon for unit sphere is an example showing that $k(L_2) = k(M_2) \leqq 2/3$ $(< 2^{-1/2})$. A tedious computation has shown that $2/3$ is precise; that is, that $k(L_2) = k(M_2) = 2/3$. So far all my attempts to show $k(L_n)$ and $k(M_n) \geqq 1/n$ have failed for $n > 2$.

The rest of this paper is devoted to extending the results of [II]. A normed vector space T of real-valued functions $t = \{t_s\}$ on some set of indices S will be called a *proper function space* if for every function $t = \{t_s\}$ in T with $0 \leqq t_s$ for all s (a) for every real-valued function $\{t_s'\}$ with $0 \leqq t_s' \leqq t_s$ for all s, the function $\{t_s'\} \in T$ and (b) $0 \leqq \|\{t_s'\}\| \leqq \|\{t_s\}\|$. If T is a proper function space and B_s, $s \in S$, are normed vector spaces, let $\mathcal{P}_T\{B_s\}$ be the space of functions $b = \{b_s\}$ where $b_s \in B_s$ and the function $\{\|b_s\|\} \in T$; in $\mathcal{P}_T\{B_s\}$, $\|b\| = \|\{b_s\}\| = \|\{\|b_s\|\}\|$. (In [II] S was countable and only the special product spaces $\mathcal{P}^p\{B_s\} = \mathcal{P}_{l^p}\{B_s\}$ were used.)

THEOREM 3. *If T is a proper function space, then $\mathcal{P}_T\{B_s\}$ is uniformly convex if and only if T is uniformly convex and the spaces B_s have a common modulus of convexity.*

As the proof follows the lines of the proof of Theorem 3 of [II] except at one point it suffices to give the first half of the sufficiency proof; that is, the special case in which $\|b\| = \|b'\| = 1$, $\|b - b'\| \geqq \epsilon$ and $\|b_s\| = \|b_s'\|$ for every s. Let $\beta_s = \|b_s\|$ and $\gamma_s = \|b_s - b_s'\|$; then for each s, $\|b_s + b_s'\| \leqq 2(1 - \delta(\gamma_s/\beta_s))\beta_s$ where δ is a common modulus of convexity for all B_s. Hence

$$(1) \qquad \|b + b'\| = \|\{\|b_s + b_s'\|\}\|_T \leqq 2\|\{1 - \delta(\gamma_s/\beta_s)\beta_s\}\|_T.$$

Clearly $\gamma_s \leqq 2\beta_s$ for all s; let E be the set of all s for which $\gamma_s/\beta_s > \epsilon/4$; then in F, the complement of E, $\beta_s \geqq 4\gamma_s/\epsilon$. If $\{t_s\}$ is any element of T, let $t_{sE} = t_s$ if $s \in E$, $t_{sE} = 0$ if $s \notin E$; then

$$1 \geqq \|\{\beta_s\}\|_T \geqq \|\{\beta_{sF}\}\| \geqq \|\{4\gamma_{sF}/\epsilon\}\| = (4/\epsilon)\|\{\gamma_{sF}\}\|.$$

Hence $\|\{\gamma_{\alpha F}\}\| \leqq \epsilon/4$ and

$$\|\{\gamma_{\alpha E}\}\| = \|\{\gamma_\alpha\} - \{\gamma_{\alpha F}\}\| \geqq \|\{\gamma_\alpha\}\| - \|\{\gamma_{\alpha F}\}\| \geqq 3\epsilon/4.$$

Hence $\|\{\beta_{\alpha E}\}\| \geqq \|\{\gamma_{\alpha E}\}\|/2 \geqq 3\epsilon/8$.

Now let $t = \{\beta_{\alpha F}\}$, $t' = \{\beta_{\alpha E}\}$ and $t'' = (1-2\delta(\epsilon/4))t'$; then $\|t+t''\| \leqq \|t+t'\| = 1$ and $\|t+t'-(t+t'')\| = \|t'-t''\| = 2\delta(\epsilon/4)\|t'\| \geqq 3\delta(\epsilon/4)\epsilon/4$. Call this last quantity $\alpha(\epsilon)$; then

$$(2) \quad \|(1-\delta(\epsilon/4))t' + t\| = (1/2)\|t+t'+t+t''\| \leqq 1 - \delta_1(\alpha(\epsilon))$$

where δ_1 is the function which exists in T by Lemma 1. By (1) and (2)

$$\|b+b'\| \leqq \|\{(1-\delta(\gamma_\alpha/\beta_\alpha))\beta_{\alpha E}\} + \{\beta_{\alpha F}\}\| \leqq \|(1-\delta(\epsilon/4))t' + t\|$$
$$\leqq 1 - \delta_1(\alpha(\epsilon)) \equiv 1 - \delta_2(\epsilon).$$

The remainder of the proof is exactly that given in [II] (beginning with line 4 on p. 506); it shows that a suitable value of δ_3 in $\mathcal{P}_T\{B_\alpha\}$ is given if $\delta_3(\epsilon) = \delta_1(\eta)$ where η is so chosen that $\eta/2 + \delta_1(\eta) < \delta_2(\epsilon)$. Since δ_3 depends only on the moduli of convexity in T and all B_α, we have the following result, more general than Corollary 1 of [II].

COROLLARY. *If $\{T\}$ is a collection of proper function spaces, if $\{B\}$ is a collection of normed vector spaces, and if all these spaces have a common modulus of convexity, then all the spaces $\mathcal{P}_T\{B_\alpha\}$ with T in $\{T\}$ and all B_α in $\{B\}$ have a common modulus of convexity.*

Some extensions of Theorem 3 may be made; for instance, it is clear that the condition (a) on a proper function space is imposed to make sure that such functions as $\{\|b_\alpha + b'_\alpha\|\}$ are in T. For example, if S is a space in which a measure is defined and all B_α are the same space B_0, it suffices to take $T = L^p_B$, $1 < p < \infty$ and to consider only Bochner measurable functions[4] $\{b_\alpha\}$ for which $\{\|b_\alpha\|\} \in T$. In this case all the functions constructed are again in T so the proof can be carried through showing directly that $L^p(B_0)$ is uniformly convex if $1 < p < \infty$ and B_0 is uniformly convex. In fact, if the norm in T satisfies (b) and if it is assumed only that every measurable real-valued function dominated by a function in T is again in T, the proof can be carried through for the space of Bochner measurable functions from S into B for which $\{\|b_\alpha\|\} \in T$.

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[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. vol. 20 (1933) pp. 262–276.

THE RADICAL OF A GROUP WITH OPERATORS

GARRETT BIRKHOFF

In a recent note (*The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 893–897) A. A. Albert has defined the radical of a general linear algebra, and deduced some of its properties from his theory of isotopy. The purpose of the present note is to extend this concept to groups with operators, and to derive similar properties from lattice theory.

Let G be a group with a class Ω of endomorphisms including all inner automorphisms.[1] By an *ideal*, we mean a subgroup S of G such that $s \in S$ and $w \in \Omega$ imply $sw \in S$. Clearly G and the group identity 0 are ideals; any other ideal is called a *proper ideal*. We recall that the ideals of G form a *modular lattice*;[2] we shall assume below that this has *finite length*.

We define G to be *prime* if and only if it has no proper ideals; it is well known (and easy to prove) that G/S is prime if and only if S is maximal. Now let $\mathfrak{Z}$ be *any* class of groups with operators (specializing to zero algebras) which is invariant under isomorphisms,[3] and let us define a *simple* group (with operators) to be any prime group not in the exceptional class $\mathfrak{Z}$. We define a direct sum of prime groups to be *semiprime*, and (following Albert and others) a direct sum of simple groups to be *semisimple*.

We further define the *ϕ-ideal* Φ of G as the intersection of all maximal ideals (by analogy with the ϕ-subgroup of a group), the *radical* R of G as the intersection of all maximal ideals S such that G/S is simple, and denote by Z the intersection of all maximal ideals T such that $(G/T) \in \mathfrak{Z}$.

THEOREM 1. *The quotient group G/S is semiprime if and only if S contains Φ.*

PROOF. If G/S is semiprime, then it can be written as a direct sum $G/S = (S_1/S) + \cdots + (S_r/S)$, where the (S_k/S) are prime. The $B_i = S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_r$ are maximal ideals; and

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[1] If G is commutative, Ω may be void. The case of a linear algebra is included by requiring Ω to consist of all scalar multiplications $x \to \lambda x$, all left multiplications $x \to ax$, and all right multiplications $x \to xa$. It follows that ideals and direct sums correspond.

[2] In the sense of the author's *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940. This will be referred to below as [LT].

[3] It is to be observed that a zero algebra is *not* a "zero" group with operators: $\lambda x \neq 0$ for scalar operators λ.

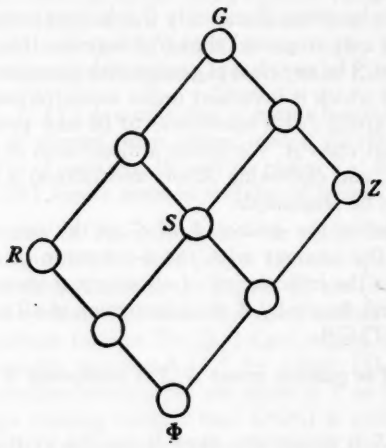
$$S = \bigwedge_{i=1}^{r} B_i \geqq \Phi.$$

To prove the converse, note that Φ is the meet of dual points (in the sense of lattice theory), and hence (by [LT, Theorem 4.1]) the modular lattice of ideals X satisfying $\Phi \leqq X \leqq G$ is *complemented.* It follows (ibid, §65) that we can find ideals $D_1, \cdots, D_r$ prime over X such that

$$(D_1 \cup \cdots \cup D_{k-1}) \cap D_k = X \text{ for all } k,$$

whence $G/X = (D_1/X) + \cdots + (D_r/X)$, in the sense of being a direct sum.

THEOREM 2. *The quotient group G/S is semisimple if and only if S contains R*



PROOF. We consider the complemented modular lattice just described. "Projective" quotients (in the sense of lattice theory) are operator-isomorphic [LT, Theorem 3.16]; hence no S for which G/S is simple is projective with an S for which G/S is in $\mathfrak{Z}$. It follows [LT, Theorem 4.6 and Corollary] that R/Φ and Z/Φ are "neutral" complements in G/Φ. Consequently any S such that G/S is semiprime (let alone semisimple) fits into the lattice just described according to the scheme indicated graphically above. We have, by well known theorems, the isomorphisms

$$G/S \cong (R \cup S/S) + (Z \cup S/S),$$
$$S/\Phi \cong (R \cap S/\Phi) + (Z \cap S/\Phi),$$

where $R \cup S/S \cong R/R \cap S \cong G/Z \cup S$ is a direct sum of groups in $\mathfrak{Z}$ and $Z \cup S/S \cong Z/Z \cap S \cong G/R \cup S$ is semisimple. We omit the details, which are identical with those of the second part of the proof of Theorem 1 as regards representation as direct sums of prime groups, and involve [LT, Theorem 4.5], for the basic isomorphisms. Theorem 2 appears as an obvious corollary of these facts and Theorem 1.

The remarkable fact about the radical of a linear associative algebra from this point of view is that it is *nilpotent*. Also, the radical of a complex Lie algebra A is *solvable*, and $[A, A]$ is *nilpotent*. The radical of an alternative algebra is also nilpotent. The generalization of these properties to more general classes of algebras would be extremely important.

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MINIMAL A-SETS, INFINITE ORBITS, AND FIXED ELEMENTS

G. E. SCHWEIGERT

Throughout this note S denotes a semi-locally-connected continuum[1] and $T(S)=S$ an onto-homeomorphism. If E is a set of S such that $T^k(E)=E$ for some positive integer k, then E has a *finite period*, otherwise the period for E is *infinite*. The set of all images of E under T and its inverse T^{-1} is said to be the *orbit* of E. If E is of period $k=1$, then E is *invariant* under T. We investigate the least invariant A-set which contains the orbit of a cyclic element E, when the period and orbit for E are infinite. This work occupies an intermediate position between a previous paper,[2] wherein certain general results are directed toward the study of finite orbits, and the problem of the action of $T(S)=S$ in general. We follow here the spirit of the work of Ayres begun in his paper, *On transformations having periodic properties*, Fund. Math. vol. 33 (1939) pp. 95–103. With Ayres we denote the unique cyclic chain between two cyclic elements E and D by $C(E, D)$.

THEOREM[3] A. *If E is any cyclic element of S with an infinite period and B is the least (invariant[4]) A-set containing the orbit of E then one of the following cases must occur:*

(a) *B contains exactly one fixed element F. In this case if B is cyclic then E is a single point which is a cut point of S lying in B. The set $S-B$ then has infinitely many distinct components bounded by images of E. (Of course it may also have other components not bounded by these images.) If B is not cyclic then for any element E' of the orbit of E the set B is the closure of the orbit of the cyclic chain $C(E', F)$.*

(b) *B contains exactly two fixed elements X and Y. In this case X*

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, pp. 64–98. It is assumed that the reader is familiar with the cyclic element theory and the general terminology of this publication. We refer to this book hereafter as ATW with the numbers for theorems in parentheses.

[2] *Fixed elements and periodic types for homeomorphism on s.l.c. continua*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1942) p. 52. To appear in full in the Amer. J. Math. under the same title. This paper will be referred to as F; the numbers in parentheses refer to theorems of the complete paper.

[3] Compare F(2.4). Each theorem of this paper has as a corollary the special case in which S is a dendrite. The form used in stating Theorem A and the general nature of the proof are at the suggestion of the referee.

[4] The least A-set containing an invariant set is invariant.

and Y are end points of B and all elements of B except X and Y have infinite periods. Furthermore:

(1) *if $B=C(X, Y)$ then the images $T^n(E)$ are linearly ordered on $C(X, Y)$ as they occur under T for n in the range of all integers.*

(2) *if $B\neq C(X, Y)$ then $C(X, Y)$ contains an element E^* satisfying* (1) *and such that $C(X, Y)$ is the minimal invariant A-set in S containing the orbit of E^*. Moreover, if E_n and E_n^* denote the nth images of E and E^*, respectively, under T, then the cyclic chain $C(E_n, E_n^*)$ in B contains the component of $B-E_n^*$ which has E_n as a subset of its closure.*

PROOF. By a theorem of Ayres[5] B must contain at least one invariant cyclic element. If it contains exactly one such invariant element F then (a) holds and the results in this case are a very easy exercise for the reader. Otherwise, B must contain at least two invariant cyclic elements and we denote any two such elements by X and Y. Thus the cyclic chain $C(X, Y)$ is invariant under T and every element which is not fixed has an infinite orbit. Now if $C(X, Y)$ contains an element of the orbit of E then it must contain the entire orbit and we have $B=C(X, Y)$. It follows at once that $C(X, Y)$ can contain no other invariant element by the fact that B is minimal and X and Y were arbitrary. From the minimal character of B we see that X and Y are end points of B, and (1) holds. In the event that no element in the orbit of E is in B there is again no third invariant element in B. Such an element Z would imply that $Z\subset\overline{K}$, where K is some component of $B-C(X, Y)$. From the minimal property of B it follows that $\overline{K}$ must contain an element of the orbit of E. Thus the entire orbit of E must lie in both of the S cyclic chains $C(Z, X)$ and $C(Z, Y)$. This easily gives an A-set which contains the orbit of E and is a proper subset of B, contrary to the definition of this set. Thus B contains exactly two invariant cyclic elements X and Y. Let $K(E)$ denote the component of $B-C(X, Y)$ having E as a subset of its closure, and p the boundary point of this component. Then if E^* is the minimal cyclic element of $C(X, Y)$ containing p we see that E^* has an infinite orbit. This gives us (2) without difficulty and completes the proof of Theorem A.

The property mentioned below is one of five previously shown[6] to be equivalent.

COROLLARY. *Let E be any cyclic element in S such that E and $E_1=T(E)$ are distinct. In order to insure that $C(E, E_1)$ has the third property of Ayres—contains exactly one invariant element Z—it is suffi-*

[5] ATW (2.51) p. 242.

[6] F (2.6).

cient (and necessary) to assume that an invariant element in $C(E, E_1)$ exists provided E has an infinite period. There is no assumption when E has finite period.

PROOF. If the period for E is infinite this follows from Theorem A. If E has finite period, the least A-set M containing the orbit of E also contains a fixed cyclic element Z by virtue of the theorem of Ayres. It is readily shown that: (1) M is the finite orbit of $C(E, E_1)$, (2) $Z \subset C(E, E_1)$, (3) Z is unique.

If M is a sum of cyclic elements of S and each element of M has a finite period then T is said to be *elementwise periodic* on M. The case $M = S - L$, where L is the set of all end points of S, has received considerable attention;[7] however, the question treated below awaited a study of infinite orbits.

THEOREM B. *If $T(S) = S$ is elementwise periodic on $S - L$ and $p \in L$ has an infinite orbit, then the least A-set M containing the orbit of p has a unique invariant element ("hub") E, and the cyclic element hyperspace of M is structurally similar to an n-adic tree.*[8]

PROOF. The set M is invariant and contains the closure of the orbit of p. In view of the density of the elements of finite period and Theorem A it is evident that M contains a unique invariant element E. Let k be the least positive value of the period function on $M - E$ and $I(k)$ the set of all cyclic elements of M fixed under T^k. It follows[9] that $I(k)$ is an A-set. Furthermore the minimal property for M insures that $M - E$ is a component-orbit, the finite orbit of some component K of $M - E$. We select one particular K and consider $\overline{K} = K + q$, with q a cut point of M on E. (Note: $T(q) \neq q$ although $T^2(q) = q$ is possible.) Let $q \in C$, where C is a true cyclic element of K, then $q \in I(k)$ implies $C \subset I(k)$. If C does not exist, q is an end point of K, and hence, by a previous theorem,[10] we again get $I(k) \cdot K \neq 0$. Thus, in any case, there is a node X of $I(k)$ in $\overline{K}$ by virtue of well known results.[11] Finally $C(X, E) \subset I(k)$ also follows from a previous theorem.[12] We now let $N = I(k) \cdot \overline{K}$ and investigate the nature of this A-set and its complement. If R is a component of $\overline{K} - N$, $F(R) = x$,

[7] F (4.2) and ATW (4.6) p. 248.

[8] Tree = dendrite of ATW. For the dyadic tree see *A note on the limit of orbits*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 968. Also see Leo Zippin, *Transformation groups*, in *Lectures in topology*, University of Michigan Press, 1941, p. 196.

[9] ATW (4.4) p. 248 adapted in F (2.6).

[10] F (2.1).

[11] ATW (8.1) and (8.2) p. 77.

[12] ATW (4.3) p. 248 adapted in F (2.6).

and E_x the cyclic element of N which contains x, then $T^k(R)\cdot R=0$ and $T^k(x)\in E_x$; for otherwise $T^k(R)=R$ and $I(k)\cdot R\neq 0$ by the argument used to establish X above. Thus $\overline{K}-N$ is a component-orbit under T^k and each component in this finite orbit is bounded by a point of E_x. If $x\notin X$ then it is possible to construct an A-set M^* which is a proper subset of M and contains the orbit of p. This would be contrary to the definition of M. A similar set M^* is possible if there is a node of $I(k)$, other than X, in N. Hence $N+E=C(X, E)=C(X, q)+E$ is the only possibility. Moreover each component of $\overline{K}-N$ has as its boundary a point of X; that is, $X=E_x$. This suffices to give the structural character of M "near" the "center of rotation" E in M; namely $C(X, q)$ and its $k-1$ images extending radially as "spokes" from the "hub" E, or a variant of this (when the period of q is less than k) in which several spokes meet in the same cut point on the orbit of q. More specifically, $I(k)=\sum_{n=1}^{k}T^n(C(X, q))+E$ and the interior of this A-set contains $I(k)-\sum T^n(X)$.

Since $I(k)$ contains no point of the orbit of p we may iterate the discussion above using $\overline{K}$ and T^k in the roles M and T. If $l>k$ is the least positive value of the period function on $K-N$ and $I(l)$ is the set of all elements in M fixed under T^l, then l is an integral multiple of k, $I(k)\subset I(l)$, and $I(l)$ is an A-set. The argument proceeds to show that, when $N^*=I(l)\cdot\overline{R}$, we get $N^*+X=C(X^*, X)$ where X^* is uniquely the node of $I(l)$ in N^*. Then $I(l)=\sum_{n=1}^{l}T^n(C(X^*, q))+E$ and this fact, together with results concerning the interior of $I(l)$, are sufficient to introduce the third stage. The third stage introduces the fourth, and so on. When these inductive steps have all been carried out, we find that (as we progress from E toward the orbit of p) the spoke $C(X, E)$ branches into l/k chains at X, these in turn branch at the images of X^*, and such branching continues indefinitely. This branching occurs only at the designated elements.

The expanding approximation due to Whyburn[13] (and adapted[14] to this type of homeomorphism) insures that the above process will envelop all but the end points of M, that is, $M-L$. But $\overline{M-L}=M$; and it can be readily shown that L is the closure of the orbit of p, hence perfect, one-dimensional, and uncountable. This ends the discussion of Theorem B.

In conclusion it is evident that the proof of Theorem B does not require the full strength of the elementwise periodicity assumption; a study of the general onto-homeomorphism and the subsequent selection of convenient restrictions will clarify this matter. It may also be

[13] ATW (4.7) p. 249.
[14] F (4.2).

remarked that Theorem A may well carry, in such a study, a weight greater than that indicated by its relatively minor role in the proof of Theorem B.

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THE EQUIVALENCE OF n-MEASURE AND LEBESGUE MEASURE IN E_n

ARTHUR SARD

Consider a set A of points in euclidean n-space E_n. For each countable covering $\{A_i\}$ of A by arbitrary sets consider the sum

$$\sigma = \sum_i c_m \delta(A_i)^m,$$

where m is a fixed positive number, $c_m = \pi^{m/2}/2^m\Gamma[(m+2)/2]$, and $\delta(A)$ is the diameter of A. The constant c_m is, for integral m, the m-volume of a sphere of unit diameter in E_m. Let $L_m(A;\alpha)$ be the greatest lower bound of all sums σ corresponding to coverings for which $\delta(A_i)<\alpha$ for all i $(\alpha>0)$. We define the m-measure of A as $L_m(A) = \lim_{\alpha\to 0} L_m(A;\alpha)$. We denote the outer Lebesgue measure of A by $|A|$.

We shall show that *n-measure and outer Lebesgue measure are equal*: $L_n(A) = |A|$. A statement on this matter by W. Hurewicz and H. Wallman is true but misleading: these authors assert that $L_n(A)/c_n$ and $|A|$ may be unequal.[1]

F. Hausdorff has introduced an m-measure $L_m^S(A)$ defined as is $L_m(A)$ except that coverings by spheres are used instead of coverings by arbitrary sets. He has shown[2] that $L_n^S(A) = |A|$. However $L_m(A)$ and $L_m^S(A)$ are unequal in general, as A. S. Besicovitch has shown[3] for $m=1$, $n=2$. S. Saks[4] and others define m-measure as $L_m(A)/c_m$.

Our proof, which is an obvious extension of Hausdorff's proof, depends on two known theorems.

THEOREM I. *Of all sets in E_n having a given diameter, the n-sphere has the greatest outer Lebesgue measure.*[5]

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[1] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 104.

[2] F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1919) p. 163.

[3] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. vol. 98 (1928) pp. 458–464. R. L. Jeffery, *Sets of k-extent in n-dimensional space*, Trans. Amer. Math. Soc. vol. 35 (1933) p. 634.

[4] S. Saks, *Theory of the integral*, Warsaw, 1937, pp. 53–54.

THEOREM II. *Suppose that to each point x of a set A in E_n there corresponds a set of closed n-spheres centered at x of arbitrarily small positive diameter. Then for any given $\epsilon>0$, a countable number of the spheres cover A and are such that the sum of their Lebesgue measures is at most $|A|+\epsilon$.*[6]

We now prove that

$$|A| \leqq L_n(A) \leqq L_n^S(A) \leqq |A|.$$

For any countable covering $\{A_i\}$ of A,

$$|A| \leqq \sum_i |A_i| \leqq \sum_i c_n\delta(A_i)^n$$

by Theorem I. Hence $|A| \leqq L_n(A;\alpha)$ for all α and $|A| \leqq L_n(A)$.

The definitions imply that $L_n(A) \leqq L_n^S(A)$.

Finally, given $\epsilon>0$ and $\alpha>0$, assign to each point x of A the set of all closed spheres centered at x and of positive diameter less than α. Then by Theorem II a countable number of these spheres $\{S_i\}$ cover A and are such that

$$\sum_i |S_i| = \sum_i c_n\delta(S_i)^n \leqq |A| + \epsilon.$$

Hence $L_n^S(A;\alpha) \leqq |A|$ and $L_n^S(A) \leqq |A|$.

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[5] W. H. and G. C. Young, *The theory of sets of points*, Cambridge, 1906, pp. 293–294. L. Bieberbach, *Über eine Extremaleigenschaft des Kreises*, Jber. Deutschen Math. Verein. vol. 24 (1915) pp. 247–250. T. Kubota, *Über die konvex-geschlossenen Mannigfaltigkeiten im n-dimensionalen Räume*, Science Reports, Tôhoku Imperial University, vol. 14 (1925) p. 98. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik vol. 3 (1934) pp. 76 and 107. W. Feller, *Some geometric inequalities*, Duke Math. J. vol. 9 (1942) pp. 889–892. The diameter of an arbitrary set B equals the diameter of the smallest closed convex set containing B.

[6] H. Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik vol. 27 (1916) p. 190. The case $|A| = \infty$ is not excluded.

A CONVERGENCE THEOREM FOR LEBESGUE-STIELTJES INTEGRALS

FUMIO YAGI

This paper deals with the following theorem.

THEOREM. *Let $g(x)$ be a bounded Borel measurable function defined everywhere on $(-\infty, \infty)$. Let $p_n(x)$ be a sequence of normalized functions,*[1] *$p_n \in V$, such that*

$$\sum_{n=1}^{\infty} V(p_n) < \infty. \tag{1}$$

Then $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent to a normalized function $p(x) \in V$, and

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{2}$$

We shall first prove this theorem in the following special cases:

(a) $g(x)$ is a bounded piecewise absolutely continuous function[2] in $(-\infty, \infty)$.

(b) $g(x)$ is continuous in a finite interval and vanishes identically outside this finite interval. (It need not necessarily be continuous at the end points of the interval.)

(c) $g(x)$ is a bounded continuous function in $(-\infty, \infty)$.

First let us prove our assertion concerning $p(x)$. Since $p_n(x)$ are normalized, we have[3] $\sum_{n=1}^{\infty} |p_n(x)| \ll \sum_{n=1}^{\infty} P_n \ll \sum_{n=1}^{\infty} V(p_n)$ where P_n is the upper bound of $|p_n(x)|$ for $-\infty < x < \infty$. Because of (1), it follows that the series $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent. Let $p(x)$ be the limit function. Evidently $p(x)$ is right-continuous and normalized. To show that $p(x) \in V$ it is sufficient to show that $p(x)$ is of bounded variation on $(-\infty, \infty)$. For $\xi > 0$ and any subdivision of $(-\xi, \xi)$, $-\xi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \xi$, we

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[1] $p(x) \in V$ means that $p(x)$ is right-continuous and of bounded variation on the infinite interval $(-\infty, \infty)$. It is normalized if $p(0) = 0$. $V(p_n)$ denotes the total variation of p_n over $(-\infty, \infty)$.

[2] $f(x)$ is piecewise absolutely continuous in $(-\infty, \infty)$ if we can divide $(-\infty, \infty)$ into a finite number of intervals such that in each of these intervals $f(x)$ is absolutely continuous.

[3] "$\ll$" is to be read "is dominated termwise by."

have

$$\sum_{i=0}^{m} \left| p(x_{i+1}) - p(x_i) \right| \leqq \sum_{n=0}^{\infty} \sum_{i=0}^{m} \left| p_n(x_{i+1}) - p_n(x_i) \right|.$$

This implies that[4] $V(p;\ -\xi,\ \xi) \leqq \sum_{n=1}^{\infty} V(p_n;\ -\xi,\ \xi) \leqq \sum_{n=1}^{\infty} V(p_n)$. Hence $V(p) < \infty$.

Let x_i $(i=1, 2, \cdots, N)$ be the points of discontinuity of $g(x)$ under condition (a), where $x_1 < x_2 < \cdots < x_N$. Then $g(x)$ is absolutely continuous on $x_i < x < x_{i+1}$ and $g(x) = \int_{x_i+0}^{x} g_1'(x)dx + g(x_i+0)$, where $g_1(x)$ is the absolutely continuous part of $g(x)$ in the Lebesgue decomposition. For the infinite interval we then have

$$\int_{-\infty}^{\infty} dg(x) = \int_{-\infty}^{\infty} g_1'(x)dx + \sum_{i=1}^{N} j(x_i)$$

where $j(x_i)$ is the jump of $g(x)$ at $x = x_i$. Since $g(x)$ is bounded, it is evident that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists. In addition we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \left| g(x) \right| \left| dp_n(x) \right| \leqq G \cdot \sum_{n=1}^{\infty} V(p_n) < \infty \tag{3}$$

where $G = \text{u.b.}_{-\infty < x < \infty} \left| g(x) \right|$. Hence $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x)$ exists.

We shall now show that (2) holds for case (a). Let A be any positive number. Then because of the above remark we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= \sum_{n=1}^{\infty} \Big\{ g(A)p_n(A) - g(-A)p_n(-A) \\ &\quad - \int_{-A}^{A} p_n(x)dg(x) \\ &= g(A)p(A) - g(-A)p(-A) \\ &\quad - \sum_{n=1}^{\infty} \int_{-A}^{A} p_n(x)g_1'(x)dx - \sum_{n=1}^{\infty} \sum_{i=1}^{N} p_n(x_i)j(x_i). \end{aligned} \tag{4}$$

By the general convergence theorem[5] of Lebesgue, this becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= g(A)p(A) - g(-A)p(-A) \\ &\quad - \int_{-A}^{A} p(x)g_1'(x)dx - \sum_{i=1}^{N} p(x_i)j(x_i) \\ &= \int_{-A}^{A} g(x)dp(x). \end{aligned} \tag{5}$$

[4] $V(p_n;\ -\xi,\ \xi)$ denotes the total variation of p_n over $(-\xi,\ \xi)$.

[5] E. C. Titchmarsh, *Theory of functions*, Oxford, 2nd ed., p. 345.

Letting $A \to \infty$, we get

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{6}$$

Now consider

$$\begin{aligned}\sum_{n=1}^{\infty} \left| \int_{-A}^{A} g(x)dp_n(x) \right| &\ll \sum_{n=1}^{\infty} \int_{-A}^{A} |g(x)||dp_n(x)| \\ &\ll G \cdot \sum_{n=1}^{\infty} \int_{-A}^{A} |dp_n(x)| \\ &\ll G \cdot \sum_{n=1}^{\infty} V(p_n).\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x)$ is absolutely and uniformly convergent in A. Hence by a known theorem,[6] we have

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x).$$

Combining this with (6), we get

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x),$$

which concludes case (a).

To prove the theorem for case (b), it is sufficient to show that the expression

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right|$$

can be made arbitrarily small for all sufficiently large values of N. Let $[M_1, M_2]$ be the interval within which $g(x)$ is continuous. By the Weierstrass approximation theorem, there exists a sequence of polynomials $g_m(x)$ which converge uniformly to $g(x)$ in $[M_1, M_2]$. Consider the functions defined by

$$f_m(x) = \begin{cases} g_m(x), & M_1 \leqq x \leqq M_2, \\ 0, & \text{otherwise.} \end{cases}$$

We note that $f_m(x)$ for $m = 1, 2, \cdots$ all satisfy the hypothesis (a). Hence we have

[6] See Hobson, *Theory of functions of a real variable*, vol. 2, p. 121.

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_m(x)dp_n(x) = \int_{-\infty}^{\infty} f_m(x)dp(x) \tag{7}$$

for $m=1, 2, \cdots$. Furthermore we have

$$\begin{aligned}\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| &\leqq \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| \\ &+ \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x) - \int_{-\infty}^{\infty} f_m(x)dp(x)\right| \\ &+ \left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right|.\end{aligned} \tag{8}$$

Because of (7) it follows that

$$\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| < \epsilon_N(m), \tag{9}$$

where $\epsilon_N(m)\to 0$ as $N\to\infty$ for every $m=1, 2, \cdots$. Since $f_m(x)$ converges uniformly to $g(x)$ in $[M_1, M_2]$ we also have

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| \\ &\quad = \left|\int_{M_1}^{M_2}\{g_m(x) - g(x)\}dp(x)\right| \leqq \epsilon_m \cdot V(p) = \epsilon_m',\end{aligned} \tag{10}$$

where $\epsilon_m = \max_{M_1\leqq x\leqq M_2}|g_m - g|$ and where $\epsilon_m'\to 0$ when $m\to\infty$ since $V(p)$ is finite. Also

$$\begin{aligned}\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| &\leqq \left|\sum_{n=1}^{N}\int_{M_1}^{M_2}\{g(x) - g_m(x)\}dp_n(x)\right| \\ &\leqq \epsilon_m \cdot \sum_{n=1}^{N}\int_{M_1}^{M_2}\left|dp_n(x)\right| \\ &\leqq \epsilon_m \cdot \sum_{n=1}^{\infty} V(p_n) = \epsilon_m''\end{aligned} \tag{11}$$

where $\epsilon_m''\to 0$ when $m\to\infty$, since $\sum_{n=1}^{\infty} V(p_n)$ is finite.

Now let δ be an arbitrary positive number and take m large enough so that

$$\epsilon_m' + \epsilon_m'' < 2\delta/3. \tag{12}$$

With m fixed, take N large enough so that

$$\epsilon_{N+\nu}(m) < \delta/3, \qquad \nu = 0, 1, 2, \cdots. \tag{13}$$

It follows that [using (12) and (13) in (8)]

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right| < \delta$$

for all sufficiently large N and arbitrary $\delta > 0$.

Coming to case (c), let α be a positive number and define

$$g_\alpha(x) = \begin{cases} g(x), & -\alpha \leqq x \leqq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g_\alpha(x)$ satisfies the conditions of hypothesis (b), we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) = \int_{-\infty}^{\infty} g_\alpha(x)dp(x). \tag{14}$$

By dominated convergence we have

$$\lim_{\alpha=\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{15}$$

Let $G = \text{u.b.}_{-\infty<x<\infty} |g(x)|$. Then

$$\sum_{n=1}^{\infty} \left| \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \right| \ll \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |g(x)|\,|dp_n(x)|$$

$$\ll G \cdot \sum_{n=1}^{\infty} V(p_n).$$

Thus $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x)$ converges absolutely and uniformly in α, and it follows that

$$\begin{aligned} \lim_{\alpha=\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) &= \sum_{n=1}^{\infty} \lim_{\alpha=\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x), \end{aligned} \tag{16}$$

the last step being justified by the theorem on dominated conver-

gence. Combining (14), (15) and (16), we obtain (2). This proves the theorem under hypothesis (c).

To complete the proof of the theorem we need the following lemma.

LEMMA. *Let $p_n(x)$ satisfy the conditions of the theorem. Suppose* (2) *is true for all bounded functions $g(x)$ of Baire class less than α. Then* (2) *is true for all bounded functions $g(x)$ in the class α.*

Let $f_j(x)$ be any convergent sequence in a Baire class $\alpha_1<\alpha$ such that $f_j(x)$ is uniformly bounded. Let $\lim_{j=\infty} f_j(x)=f^*(x)$. Clearly $f^*(x)$ is also bounded. By hypothesis, we have

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x) = \int_{-\infty}^{\infty} f_j(x)dp(x) \tag{17}$$

for $j=1, 2, \cdots$. By a theorem on bounded convergence, we have

$$\lim_{j=\infty}\int_{-\infty}^{\infty} f_j(x)dp(x) = \int_{-\infty}^{\infty} f^*(x)dp(x). \tag{18}$$

Consider also

$$\sum_{n=1}^{\infty}\left|\int_{-\infty}^{\infty} f_j(x)dp_n(x)\right| \ll F\cdot\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}|dp_n(x)| = F\cdot\sum_{n=1}^{\infty} V(p_n),$$

where F is $\text{u.b.}_{-\infty<x<\infty,\ j=1,2,\cdots}|f_j(x)|$. Hence $\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x)$ converges absolutely and uniformly in j. Therefore

$$\lim_{j=\infty}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x) = \sum_{n=1}^{\infty}\lim_{j=\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f^*(x)dp_n(x). \tag{19}$$

Combining equations (17), (18) and (19), we get

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f^*(x)dp_n(x) = \int_{-\infty}^{\infty} f^*(x)dp(x),$$

which completes the proof of the lemma.

Finally we return to the original hypothesis. It is known[7] that the

[7] For example, see de la Vallée Poussin, *Intégrales de Lebesgue* (Borel Monograph), 1916, pp. 36, 37.

set of bounded functions in classes 0, 1, 2, $\cdots$ of Baire is the same as the set of bounded Borel measurable functions. Under (c) we have proved the theorem for any bounded continuous function, that is, for bounded functions in class 0 of Baire. By the lemma it holds then for functions in classes 0, 1, 2, $\cdots$ of Baire and hence for all bounded Borel measurable functions. This completes the proof of the theorem.

We note in the theorem above that the boundedness of $g(x)$ was sufficient to insure the existence of the integral $\int_{-\infty}^{\infty} g(x)dp(x)$, provided that $p(x)$ satisfied the conditions of this theorem. Now we shall prove a partial converse of this.

THEOREM. *Let $g(x)$ be a given Borel measurable function defined everywhere in $(-\infty, \infty)$, with the property that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists*[8] *whenever $p(x) \in V$. Then $g(x)$ is bounded everywhere in $(-\infty, \infty)$.*

Let E_n be the set of points x for which $n \leqq g(x) < n+1$. We shall show that the family of non-empty sets E_{n_i} is finite.

Suppose the contrary. Then there must exist an infinity of distinct x_{n_i} since E_{n_i} are mutually disjoint. Consider a particular sequence $x_{n_0}, x_{n_1}, x_{n_2}, \cdots$, one x_{n_i} from each non-empty E_{n_i}.

At this point let us note that given any sequence of integers $0 < n_1 < n_2 < \cdots$ there exists a number ξ such that $\sum_{i=1}^{\infty} 1/n_i^{\xi}$ diverges and $\sum_{i=1}^{\infty} 1/n_i^{\xi+1}$ converges. In class L put all real numbers p such that $\sum_{i=1}^{\infty} 1/n_i^{p}$ is divergent. Put all others in R-class. To the R-class belongs the number 2. The number 0 belongs to the L-class. If p belongs to L, then all numbers less than p also belong to L. Thus we have a Dedekind cut. Let c be the number defined by this cut. Then $\sum_{i=1}^{\infty} 1/n_i^{c-1/2}$ diverges, while $\sum_{i=1}^{\infty} 1/n_i^{c+1/2}$ converges. Hence $\xi = c - 1/2$ satisfies the required condition.

With this in mind, let us return to the proof of the theorem. Using the sequence x_{n_i} defined previously let us form the function

$$p(x) = \sum_{i=0}^{\infty} \frac{1}{(n_i+1)^{\xi+1}} \omega_{n_i}(x),$$

where

$$\omega_{n_i}(x) = \begin{cases} 0, & x < x_{n_i}, \\ 1, & x \geqq x_{n_i}, \end{cases}$$

and ξ is such a number that $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi+1}$ converges and $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi}$ diverges.

Since $\omega_{n_i}(x)$ is right-continuous and the series uniformly conver-

[8] M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, vol. 15, 1932, p. 206.

gent, $p(x)$ is also right-continuous. Also $p(x)$ is a monotone increasing function. To show that $p(x) \in V$, we make use of the theorem proved earlier. In fact we have

$$\int_{-\infty}^{\infty} |\, dp(x)\,| = \int_{-\infty}^{\infty} dp(x) = \int_{-\infty}^{\infty} d\left\{ \sum_{i=0}^{\infty} \frac{1}{(n_i+1)^{\xi+1}} \omega_{n_i}(x) \right\}$$

$$= \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(n_i+1)^{\xi+1}}\, d\omega_{n_i}(x)$$

$$= \sum_{i=0}^{\infty} \frac{1}{(n_i+1)^{\xi+1}}.$$

To complete the proof of this theorem it is sufficient to show that $\int_{-\infty}^{\infty} g(x)dp(x)$ does not exist for this $p(x)$. Let

$$g_M(x) = \begin{cases} g(x), & \text{if } |\, g(x)\,| \leqq M, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_{-\infty}^{\infty} |\, g_M(x)\,|\,|\, dp(x)\,| = \sum_{i=0}^{\infty} \frac{1}{(n_i+1)^{\xi+1}} \int_{-\infty}^{\infty} |\, g_M(x)\,|\, d\omega_{n_i}(x)$$

$$= \sum_{i=0}^{\infty} \frac{1}{(n_i+1)^{\xi+1}} |\, g_M(x_{n_i})\,|$$

$$\geqq \sum_{n_i<M} \frac{n_i}{(n_i+1)^{\xi+1}},$$

the first step being justified by the first theorem. Hence

$$\lim_{M=\infty} \sum_{n_i<M} \frac{n_i}{(n_i+1)^{\xi+1}} \leqq \int_{-\infty}^{\infty} |\, g(x)\,|\,|\, dp(x)\,|,$$

which proves that the integral does not exist, since the series $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi}$ was constructed to be divergent.

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DECOMPOSITIONS OF A T_1 SPACE

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Introduction. Several authors have proved theorems of the type: The "structure" of a certain class of transformations defined on a "suitable" space A to a fixed "suitable" space B "determines" the space A.

As examples, we have:

Banach [3, p. 170, see also 6, 7, 13]:[1] The Banach space of all real, continuous functions defined on a compact metric space A "determines" A.

Eidelheit [5, see also 2, 10, 11]: The ring of all bounded operators on a real Banach space A "determines" A.

In the present paper, we prove an analogous theorem (Theorem 2.5). Intuitively, it says that a T_1 space A is "determined" by a rather weak ordered system structure of the collection of all continuous mappings of A onto an arbitrary (variable) T_1 space B. More exactly, it states: If two T_1 spaces A, B are such that the ordered system of upper semi-continuous decompositions of A is isomorphic to that of B, then A and B are homeomorphic.

In §1 we give a discussion of ordered systems which is sufficient for our purposes. In §2 we prove the theorem mentioned above. In §3 we characterize separation and connectedness properties of a T_1 space in terms of order properties of its upper semi-continuous decompositions. In §4 we discuss compactness properties of the space and their relations to order properties of the decompositions, and in §5 we give some examples and counter examples.

1. Ordered systems.[2] We assume that the reader is fairly familiar with the nomenclature of ordered systems and lattices.

An *ordered system* is a collection, M, of elements, D, with an ordering defined in M. That is, there is given a binary relation, $>$, which is defined for some pairs of elements in M, and which is transitive, reflexive, and proper.

We use *less than* and *greater than* to refer to the ordering in M; *contains* will be used only in the point set sense.

Most of the ordered systems we consider will be directed and will contain atoms. Throughout the paper, we use the letters a, b, c for

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

[2] For a more complete discussion, see [4, 12, 14].

atoms of ordered systems.

A *multiplicative system* is an ordered system in which each pair of elements has an inf; we use σ-complete and complete in the usual sense.

THEOREM 1.1. *A complete multiplicative system M with a unit is a complete lattice.*

For any non-empty collection $\{D_\alpha\}$ of elements of M, define[3]

$$D' = \wedge \{D \mid D > D_\alpha, \text{ all } \alpha\}.$$

Since M has a unit, $\{D \mid D > D_\alpha, \text{ all } \alpha\}$ is not empty, and D' exists; also, from the definition of $\wedge$, $D' > D_\alpha$ for all α. Suppose $D'' > D_\alpha$ for all α. Then

$$D' = \wedge \{D \mid D > D_\alpha\} = D'' \wedge (\wedge \{D \mid D > D_\alpha, D \neq D''\} < D''$$

so that $D' = \vee D_\alpha$ and M is a complete lattice.

Two ordered systems M, M' are *isomorphic* if there is a one-to-one, order preserving correspondence between them; that is, if there exists a one-to-one mapping $f(M) = M'$ such that $D' < D''$ is equivalent to $f(D') < f(D'')$.

Some examples of ordered systems which we shall use in the sequel are:

1. The collection of all open neighborhoods of a fixed point of any topological space, ordered by point set inclusion. That is, $U < V$ means $U \subset V$. This system is directed both by $>$ and by $<$ ($<$ is the "interesting" direction), in fact, it is a lattice. It contains a unit, but usually not a zero, and even if it has a zero, it may contain no atoms.

2. The lattice of all closed subsets of a T_1 space, again ordered by point set inclusion. Since this lattice is complete, it has a zero and a unit. The closed sets containing just one point are atoms, and these are the only atoms.

2. The reconstruction of S from M. Throughout the paper, S will denote a topological space which contains an infinite number of points (four points would be sufficient), and in which each subset containing exactly one point is closed; that is, S is a T_1 space. By a decomposition of S we shall mean a collection of non-empty, closed, disjoint subsets of S whose union is S.

DEFINITION 2.1. *A decomposition of S is upper semi-continuous*

[3] We use $\vee$, $\wedge$ for sup, inf; $\cup$, $\cap$ for union, intersection, respectively; and $\{p \mid q\}$ for the set of all elements p with property q.

(u.s.c.) if, for any set X of the decomposition and any open set $U \supset X$, there exists an open set $V \supset X$ such that any set of the decomposition which meets V is contained in U.

It is well known that every upper semi-continuous decomposition (u.s.c.d.; the same abreviation will be used for the plural) of S gives rise to a continuous mapping of S onto a T_1 space—the Zerlegungsraum of Alexandroff and Hopf [1, p. 61 ff.]. Thus any statement concerning u.s.c.d. may be interpreted as a statement about continuous mappings of S onto a T_1 space.

Let $M = \{D\}$ be the ordered system of u.s.c.d. of S, where $D' < D''$ means each set of the decomposition D' is contained in some set of the decomposition D''. It is easy to see that M has a zero, 0, (the decomposition of S into the single points of S) and a unit, I, (the decomposition of S into the single set S) and is therefore directed both by $>$ and by $<$. However (Example 5.3), M is in general not a multiplicative system.

If x_a, y_a are any two distinct points of S, the decomposition of S which consists of the set $x_a \cup y_a$ and the single points of $S-(x_a \cup y_a)$ is u.s.c. and is an atom of M. Conversely, any atom of M has this form. We shall say that a is *generated* by x_a, y_a; the generators of an atom are unique, and completely determine the atom. Even though M is not necessarily a lattice, it is clear that $\bigvee a_i$ $(i=1, 2, \cdots, n)$ exists for any finite number of atoms and $a_i \wedge a_j = 0$ if $a_i \neq a_j$.

DEFINITION 2.2. *Two atoms a, b are concurrent, in symbols, $a \sim b$, if and only if it is false that $a \vee b$ is greater than exactly two atoms.*

THEOREM 2.1. *A necessary and sufficient condition that $a \sim b$ is that $(x_a \cup y_a) \cap (x_b \cup y_b) \neq 0$ (a; b generated by x_a, y_a; x_b, y_b respectively).*

NOTE. From this theorem, it will be observed that the relation $\sim$ is reflexive and symmetric but not, in general, transitive.

Sufficiency. We may assume $x_a = x_b$. Then either $y_a = y_b$ so that $a = b = a \vee b$ and $a \vee b$ is greater than just one atom, or $y_a \neq y_b$. In the latter case, $a \vee b$ is the decomposition of S into $x_a \cup y_a \cup y_b$ and the single points of $S-(x_a \cup y_a \cup y_b)$. But then there are three distinct atoms, namely a, b, and the atom generated by y_a, y_b, which are less than $a \vee b$.

Necessity. Suppose a and b are not concurrent, then $a \neq b$ since otherwise $a \vee b = a$ and is greater than just one atom. But we have seen above that if $a \neq b$ and $(x_a \cup y_a) \cap (x_b \cup y_b) \neq 0$, there are three atoms less than $a \vee b$. Therefore, since $a \vee b$ is greater than only two atoms

by hypothesis, we must have

$$(x_a \cup y_a) \cap (x_b \cup y_b) = 0.$$

DEFINITION 2.3. *An u.s.c.d., D, of S is trivial if, for any two atoms a, b less than D, there exists an atom c less than D satisfying $a \sim c$, $b \sim c$.*

THEOREM 2.2. *An u.s.c.d., D, of S is trivial if and only if it consists of single points of S except for one closed set X (which may be empty, but cannot contain just one point). We say that D is generated by X, and use the notation D_X for it.*

Sufficiency. We may suppose $X \neq 0$, since otherwise $D = 0$ and is trivial. If a, b are any two atoms less than D, the atom c generated by x_a, y_b (which will be distinct points under suitable labeling) satisfies our requirements.

Necessity. If there are two sets X, Y of the decomposition D, each containing at least two points, say $x_a, y_a \in X$, $x_b, y_b \in Y$, then a, b are two atoms less than D such that no atom c less than D satisfies $a \sim c$, $b \sim c$.

DEFINITION 2.4. *An infinite collection of atoms of M is a C-set if each pair of atoms of the collection are concurrent.*

Let us order the C-sets of M by inclusion, that is $C' > C''$ if each atom in C'' is also in C'. Then by Zorn's Lemma (or see below) there exist maximal C-sets. We denote the collection of all maximal C-sets of M by S'.

THEOREM 2.3. *There is a one-to-one correspondence between the points of S and the maximal C-sets of S'.*

Let x be any point of S and define $f(x)$ as the collection of all atoms generated by x, y for arbitrary $y \in S - x$. Then, by Theorem 2.1, $f(x)$ is a C-set. (It is infinite since S has an infinite number of points.) Moreover, it is a maximal C-set, for if C' is any C-set containing $f(x)$, and $c' \in C'$, then (Theorem 2.1) one of the generators of c' must be x, and $c' \in f(x)$. Thus $f(x)$ is a single-valued transformation of S into S'.

The inverse function f^{-1} is single-valued. For, if $f(x) = f(y)$, then any atom of M which has x as a generator also has y as a generator, and $x = y$.

$f(S)$ covers S'. For, if C is any maximal C-set, and a, b are any two distinct atoms in C, then $a \sim b$ and we may suppose $x_a = x_b$, $y_a \neq y_b$. Now, if c is any atom in C, we must have

$$(x_a \cup y_a) \cap (x_c \cup y_c) \neq 0 \neq (x_b \cup y_b) \cap (x_c \cup y_c)$$

so that (with suitable labeling) either (1) $x_c = x_a = x_b$, or (2) $x_c = y_a$ and $y_c = y_b$. But the latter case is impossible because C must contain an infinite number of atoms whereas if (2) holds no atom of $M-(a \cup b \cup c)$ can be concurrent with each of a, b, c. Thus (1) is the case, and $f(x_a) = C$.

THEOREM 2.4. *If M is the ordered system of all u.s.c.d. of a T_1 space S, then, using only the order properties of M, a T_1 space S' may be defined which is homeomorphic to S.*

The points of S' will be the maximal C-sets of M. A subset, Q, of S' will be closed if either (1) Q consists of exactly one point of S', or (2) there exists a trivial decomposition, D, of S such that Q is the collection of all maximal C-sets which contain at least one atom less than D.

It is easy to see that S' is a T_1 space. We prove that the transformation $f(S) = S'$ defined in the proof of Theorem 2.3 is a homeomorphism. We must only show that f and f^{-1} are closed.

Let X be any closed subset of S which contains more than one point. Then the closed subset of S' given by condition (2) above with $D = D_X$ is exactly $f(X)$.

Conversely, if X' is closed in S' and contains more than one point, and if D_X is the trivial decomposition given by condition (2) above, then $f^{-1}(X') = X$.

Since both S and S' are T_1 spaces, f and f^{-1} are closed and f is a homeomorphism.

THEOREM 2.5. *A necessary and sufficient condition that two T_1 spaces be homeomorphic is that their ordered systems of u.s.c.d. be isomorphic.*

The proof is immediate from Theorem 2.4.

3. Separation properties of S.

THEOREM 3.1. *If S is normal, M is a multiplicative system.*

Let $D' = \{X\}$, $D'' = \{Y\}$ be any two elements of M, and define

$$D = \{X \cap Y \mid X \in D',\ Y \in D'',\ X \cap Y \neq 0\}.$$

Clearly D is a decomposition of S; we show that it is u.s.c. If $X \cap Y \in D$ and U is any open set containing $X \cap Y$, then $X - U$, $Y - U$ are closed, disjoint subsets of S so that there exist open, disjoint subsets W', W'' containing them, respectively. Now $U \cup W' \supset X$, $U \cup W'' \supset Y$ and, since D', D'' are u.s.c., there exist open sets $V' \supset X$, $V'' \supset Y$ such that any set of $D^{(i)}$ which meets $V^{(i)}$ is contained in $U \cup W^{(i)}$ $(i = ', '')$.

Set $V=V'\cap V''$. Then, if $X'\cap Y'$ is any set of D which meets V, we have

$$X'\cap V'\neq 0, \quad \text{and} \quad X'\subset U\cup W',$$
$$Y'\cap V''\neq 0, \quad \text{and} \quad Y'\subset U\cup W'',$$

so that

$$X'\cap Y'\subset (U\cup W')\cap (U\cup W'')=U,$$

and D is u.s.c. Clearly $D=D'\wedge D''$. Hence, M is a multiplicative system.

The converse of Theorem 3.1 does not hold. See Example 5.2.

DEFINITION 3.1. *Two trivial u.s.c.d. of S, D', D'' are concurrent, in symbols $D'\sim D''$, if some atom less than D' is concurrent with some atom less than D''.*

Notice that the trivial decomposition zero is not concurrent with any decomposition, and if D', D'' are atoms, Definition 3.1 agrees with Definition 2.2.

THEOREM 3.2. *A necessary and sufficient condition that two nonzero trivial decompositions D_X, D_Y be concurrent is that $X\cap Y\neq 0$.*

Sufficiency. Let $x\in X\cap Y$, $y\in Y-x$, $z\in X-x$. The atoms generated by x, z; x, y are less than D_X, D_Y respectively, and are concurrent, so that $D_X\sim D_Y$.

Necessity. If $X\cap Y=0$, then no atom less than D_X can be concurrent with any atom less than D_Y, so that D_X and D_Y are not concurrent.

DEFINITION 3.2. *Two nonzero, trivial decompositions D', D'' run over M if each atom of M is concurrent with at least one of D', D''.*

THEOREM 3.3. *A necessary and sufficient condition that two nonzero, trivial decompositions, D_X, D_Y run over M is that $S-(X\cup Y)$ contains at most one point.*

If $S-(X\cup Y)$ contains two distinct points x, y, then the atom generated by x, y is not concurrent with either D_X or D_Y.

If $S-(X\cup Y)$ contains zero or one points, then any atom of M must have one of its generators in $X\cup Y$ and, by Theorem 3.2, must be concurrent with either D_X or D_Y.

THEOREM 3.4. *S is a Hausdorff space if and only if, for any atom $a\in M$, there exist trivial decompositions D', D'' which run over M, such that*

$$D'\sim a,\ a\sim D'',\ D'\not> a,\ a\not< D''.$$

Suppose S is a Hausdorff space, and let a be any atom of M. Then x_a, y_a have disjoint neighborhoods U, V, and $X=S-U$, $Y=S-V$ form a closed covering of S. The trivial decompositions D_X, D_Y satisfy the conditions of the theorem.

Conversely, if x_a, y_a are any two distinct points of S, let a be generated by x_a, y_a, and let D_X, D_Y be the trivial decompositions satisfying the conditions of the theorem. Then (under suitable labeling)

$$x_a \in X \cap (S - Y), \qquad y_a \in Y \cap (S - X)$$

and $U=S-X$, $V=S-Y$ are neighborhoods of x_a, y_a respectively. If $S-(X \cup Y)$ is empty, $U \cap V=0$, if $S-(X \cup Y)$ contains just one point, z, then z is an isolated point and $U-z$, $V-z$ are disjoint neighborhoods of x_a, y_a, so that S is a Hausdorff space.

The proofs of the following theorems are similar to the proof of Theorem 3.4, and will not be given here.

THEOREM 3.5. *S is regular if and only if, for any nonzero trivial decomposition D, and any atom a not concurrent with D, there exist trivial decompositions D', D'' which run over M and such that $D'>a$, a is not concurrent with D'', D' is not concurrent with D, $D<D''$.*

THEOREM 3.6. *S is normal if and only if, for any two nonzero trivial decompositions D', D'', such that D' and D'' are not concurrent, there exist trivial decompositions D''', D'''' which run over M such that $D'''>D'$, D' is not concurrent with D'''', D''' is not concurrent with D'', $D''<D''''$.*

THEOREM 3.7. *S has at least one isolated point if and only if one of the following two equivalent conditions is fulfilled.*

(I) *There exists a trivial decomposition D, and an atom $a \not< D$ which run over M.*

(II) *There exists a nonzero trivial decomposition D and an atom $a \not< D$ such that, for any atom b, $b \not< D$ implies $b \sim a$.*

THEOREM 3.8. *S is connected, except possibly for one isolated point, if and only if every two trivial decompositions which run over M are concurrent.*

4. Compactness properties of S.

THEOREM 4.1. *If S is a compact[4] Hausdorff space, then M is a complete lattice.*

[4] A topological space is compact if every open covering has a finite subcovering or, equivalently, if every collection of closed subsets whose intersection is empty contains a finite subcollection with empty intersection.

By Theorem 1.1, we need only show that M is a complete multiplicative system. Let $\{D_\alpha\}$ be any non-empty collection of u.s.c.d. of S and set

$$D = \left\{ \bigcap_\alpha X_\alpha \,\middle|\, X_\alpha \in D_\alpha, \bigcap_\alpha X_\alpha \neq 0 \right\}.$$

Clearly D is a decomposition of S; we show that it is u.s.c. If $X = \bigcap X_\alpha$ is any set of D and U is any open set containing X, then each of the sets $X_\alpha - U$ is closed and their intersection is empty. Since S is compact, there must be a finite number of the sets $X_\alpha - U$ whose intersection is empty, say

$$\bigcap_{i=1}^{n} (X_{\alpha_i} - U) = 0.$$

But S is normal, hence there exist open sets $W_i \supset X_{\alpha_i} - U$ ($i = 1, 2, \cdots, n$) such that $\bigcap W_i = 0$. Now for each i, $U \cup W_i$ is an open set containing X_{α_i} and, since each D_α is u.s.c., there exist open sets $V_i \supset X_{\alpha_i}$ such that any set of D_{α_i} which meets V_i is contained in $U \cup W_i$. Set $V = \bigcap V_i$. Then $V \supset X$, and any set of D which meets V is contained in U. For, if $X' = \bigcap X'_\alpha$ meets V, then $X'_{\alpha_i} \cap V_i \neq 0$, and $X'_{\alpha_i} \subset U \cup W_i$, $i = 1, 2, \cdots, n$, so that

$$\bigcap_\alpha X'_\alpha \subset \bigcap_{i=1}^{n} X'_{\alpha_i} \subset \bigcap_{i=1}^{n} (U \cup W_i) = U.$$

Thus D is an u.s.c.d. of S. Clearly $D = \wedge D_\alpha$, and M is a complete lattice.

THEOREM 4.2. *If S is regular, and each set of the u.s.c.d. $D = \{X\}$ is compact, then $D \wedge D'$ exists in M for any $D' = \{Y\} \in M$.*

In the proof of Theorem 3.1, the only use which we made of the normality of S was in concluding that, since $X - U$, $Y - U$ were closed, disjoint subsets of S, they could be separated by disjoint open sets W', W''. Suppose X is compact and S is regular, then, for each point $x \in X - U$, there exist disjoint open sets V_x, W_x containing x, $Y - U$ respectively. Since X is compact, a finite number, say V_{x_i} ($i = 1, \cdots, n$) of the V_x cover $X - U$ and

$$W' = \bigcup_i V_{x_i}, \qquad W'' = \bigcap_i W_{x_i}$$

are disjoint open sets containing $X - U$, $Y - U$ respectively.

THEOREM 4.3. *If S is a Hausdorff space and D', D'' are two u.s.c.d.*

of S such that each set of either decomposition is compact, then $D' \wedge D''$ exists in M.

The proof is similar to that for Theorem 4.2 and will not be given.

The author believes that the compactness hypotheses in Theorems 4.2, 4.3 are necessary, but has been unable to construct an example to prove their necessity. However, Example 5.3 shows that the separation hypotheses cannot be dropped.

THEOREM 4.4. *If S is metric, then a necessary and sufficient condition that M be a complete lattice is that S be either discrete or compact.*

By Theorem 4.1, if S is compact metric, M is a complete lattice; if S is discrete metric, then any decomposition of S is u.s.c. and M is a complete multiplicative system, hence, by Theorem 1.1, a complete lattice.

If S is metric but neither discrete nor compact, there exist two sequences of distinct points, x_i, y_i such that $\lim x_i = x$, $\{y_i\}$ has no limit point. Then the set

$$Y = x \cup (\cup x_i) \cup (\cup y_i)$$

is closed. Let a_i be the atom of M generated by x_i, y_i, then $D = \vee a_i$ does not exist in M.

For suppose $D \in M$. Consider the u.s.c.d. D_n $(n = 1, 2, \cdots)$ consisting of the sets (1) single points of $S - Y$, (2) the n sets $x_1 \cup y_1$, $x_2 \cup y_2$, $\cdots$, $x_n \cup y_n$, (3) the set $x \cup (\cup_{i>n} x_i) \cup (\cup_{i>n} y_i)$. Each $D_n > a_i$ for all i, hence $D < D_n$ for all n. Thus the decomposition D must have the single point x as one of its sets. But then D is not u.s.c. since each neighborhood of x meets all but a finite number of the sets $x_i \cup y_i$ of D, whereas the neighborhood $S - (\cup y_i)$ of x does not contain any of the sets $x_i \cup y_i$.

In this connection, see also Examples 5.1, 5.2, 5.4.

5. **Examples.** In this section we give several examples which have been referred to in the other sections of the paper. No proofs will be given.

EXAMPLE 5.1. S is the real line. M is not a σ-complete multiplicative system.

For each $n > 1$, let

$$D_n = \left\{ 1/2 \cup 2,\ 1/3 \cup 3,\ \cdots,\ 1/n \cup n,\ \left(\cup_{i>n} 1/i\right) \cup 0 \cup \left(\cup_{i>n} i\right), \right.$$
$$\left. \text{and single points of } S \text{ elsewhere} \right\}.$$

Then $\wedge D_n$ does not exist in M. See Theorem 4.4.

EXAMPLE 5.2. S is not discrete, not compact, not a Hausdorff space. M is a complete lattice.

Let S be the set of real numbers with "closed set" defined as (1) the whole space or (2) any countable (empty, finite, or denumerable) subset. Any decomposition of S is u.s.c. See Theorems 3.1, 4.4.

EXAMPLE 5.3. S is compact, not a Hausdorff space. M is not a multiplicative system.

Let S be the point set of the xy-plane consisting of the two line segments

$$L_i\colon\ y = i,\ |x| \leqq 1, \qquad i = 1, 2,$$

but the topology of S is *not* its relative topology. The closed subsets of S are any finite union of the sets (1), single points of S, or (2), any subset of S which is closed in the topology of the xy-plane and contains its projection on L_i ($i=1, 2$). (Intuitively, (2) says: any set which is the union of the "same" two "closed" subsets of the segments L_1, L_2.) Let

$$a_i = (-1/i, 0), \qquad b_i = (1 - 1/i, 0), \qquad i = 2, 3, \cdots,$$

$$Y = \bigcup (a_i \cup b_i) \cup (0, 0) \cup (0, 1) \cup (1, 0) \cup (1, 1),$$

and set

$$D' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 0),\ (0, 1) \cup (1, 1),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\},$$

$$D'' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 1),\ (0, 1) \cup (1, 0),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\}.$$

Then $D' \wedge D''$ does not exist in M. See Theorems 3.1, 4.1, 4.2, 4.3.

EXAMPLE 5.4. S is the xy-plane. M is not a lattice.

Let

$$Y = \bigcup_{m,n>0} (1/n, m/n),$$

and set

$$D' = \{\text{single points of } S - Y,\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 1, 3, 5, \cdots)\},$$

$$D'' = \left\{\text{single points of } (S - Y) \cup \left(\bigcup_n (1/n, 1/n)\right),\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 2, 4, 6, \cdots)\right\}.$$

Then $D' \vee D''$ does not exist in M. See Theorems 3.1, 4.1, 4.4.

In a future paper, the author will discuss an interesting related problem which was suggested to him by A. D. Wallace. Namely: Characterize intrinsically those ordered systems which are "M's" for some space S from a specified class, for example S compact Hausdorff.

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THE ADDITIVITY OF THE LEBESGUE AREA

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A triple of continuous functions $T: x^i(u^1, u^2)$, $i=1, 2, 3$, defined on a closed square Q $[0 \leqq u^1 \leqq 1, 0 \leqq u^2 \leqq 1]$ represents a surface $\mathfrak{S}$ (1.6, 1.17, 1.21).[1] If r is any closed rectangle in Q then we may speak of the triple T_r consisting of the above triple T with its range of definition restricted to r. This triple generates a surface $\mathfrak{S}(r)$. If r_1 and r_2 have no interior points in common, and $r_1+r_2=Q$, it is natural to hope that $L(\mathfrak{S})=L(\mathfrak{S}(r_1))+L(\mathfrak{S}(r_2))$ where L is the symbol used to indicate the Lebesgue area (3.13). This statement is certainly true whenever the Lebesgue area is given by the standard integral formula, since the Lebesgue integral is additive. However, in general it is false. It may be said that this note is concerned with the statement that if the triple is constant on $r_1 \cdot r_2$, the Lebesgue area is additive.

Stated in this fashion the theorem may appear to be new. Actually it is but a very special case of a fundamental problem in the theory of area. The classical conjecture is that the area is additive under the much weaker requirement that the triple be rectifiable on $r_1 \cdot r_2$. No proof of this conjecture has, to our knowledge, appeared. (In this connection see McShane [1, p. 138].)

In relation to existing literature the theorem of this note is included in a more general theorem due to Morrey [1]. In fact, the first case in Morrey's proof [1, p. 314] is essentially the theorem of this paper. A discrepancy in his argument was rectified by Radó and Reichelderfer [1] in an independently interesting discussion of stretching processes.

This result, therefore, is not new.

The point of this treatment is the utter simplicity of the stretching process here employed. The development serves to make a known result more accessible.

THEOREM 1. *If a polyhedron* $\mathfrak{P}$ (3.11, 1.21) *has a quasi-linear* (1.7) *representation* $T: x(u)$, $u \in Q$, *such that the image of the side* s $[u^1=1, 0 \leqq u^2 \leqq 1]$ *is interior to a sphere of diameter less than* ϵ *about the point* a $[a^1, a^2, a^3]$, *then there exists a polyhedron* $\mathfrak{P}^*$ *with quasi-linear representation* $T^*: x^*(u)$, $u \in Q$, *such that*

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[1] The notation and terminology of this note are largely due to Radó [1]; in fact, numbers in parentheses refer the reader to appropriate paragraphs in his paper. Numbers in brackets refer to the bibliography.

(1) $E(\mathfrak{P}^*)=E(\mathfrak{P})$.[2]
(2) $x^*(u)=a$ for $u\in s$.
(3) $\|x^*(u)-x(u)\|<\epsilon$, hence $d(\mathfrak{P}^*, \mathfrak{P})<\epsilon$ (1.1, 1.22).

PROOF. Define $x_\alpha(u^1, u^2)=x(\alpha u^1, u^2)$, $\alpha>1$. Since T is continuous we may choose α so that $\|x_\alpha(u)-x(u)\|<\epsilon$, for $u\in R_\alpha[0\leqq u^1\leqq 1/\alpha, 0\leqq u^2\leqq 1]$, and further the image of $S=Q-R_\alpha$ under T is in a sphere of diameter less than ϵ about a. The representation $x_\alpha(u)$ is certainly quasi-linear in R_α.

Define $x^*(u)$ to be equal to $x_\alpha(u)$ on R_α. Now $x^*(u)$ is a quasi-linear representation of $\mathfrak{P}$ on R_α.

We have yet to define $x^*(u)$ on the rest of Q. There is a set of intervals $I_1, \cdots, I_{n-2}$ on $[u^1=1/\alpha, 0\leqq u^2\leqq 1]$ such that $x^*(u)$ is linear on each of them and consecutive I's abut. Their end points, in order, are

$$p_0[1/\alpha, 0],\ p_1, \cdots, p_{n-2}[1/\alpha, 1].$$

Add to this sequence $p_{n-1}[1, 1]$ and $p_n[1, 0]$.

Let p be the point $[(\alpha+1)/2\alpha, -1]$ and consider the segments (p, p_k), $k=1, \cdots, n-1$. Each segment cuts $[1/\alpha\leqq u^1\leqq 1, u^2=0]$ in exactly one point q_k. Join p_k to q_{k+1}, $k=1, \cdots, n-2$. The rectangle S is now subdivided into $2n-2$ triangles whose vertices are in the set $p_0, \cdots, p_n, q_1, \cdots, q_{n-1}$.

The triple $x^*(u)$ has been defined at the vertices $p_0, \cdots, p_{n-2}$.

Let $x^*(p_{n-1})=x^*(p_n)=a$, and $x^*(q_k)=x^*(p_k)$, $k=1, \cdots, n-1$.

Since the mapping is defined at the vertices of each triangle of the subdivision there is a unique linear extension over each triangle. The extensions will match along the side two triangles may have in common, and hence we have a quasi-linear[3] extension on S. Define $x^*(u)$ over S to be this extension.

Now $T^*:x^*(u)$ is quasi-linear on Q and so represents a polyhedron $\mathfrak{P}^*$.

It is important to notice that the map of each of the triangles in S is a segment or a point and hence contributes zero area. Therefore $E(\mathfrak{P}^*)=E(\mathfrak{P})$.

Secondly, $x^*(u)=x^*(p_n)=x^*(p_{n-1})=a$ for $u\in s$.

Finally, $T^*(S)\subset T(S)+$[the segment from $T(p_{n-1})$ to a]. But $T(S)$

[2] $E(\mathfrak{P})$ is the elementary area of the polyhedron $\mathfrak{P}$, that is, the area in the ordinary geometric sense. It is not necessary, for the purpose of this note, to know that this coincides with the Lebesgue area. The fact that this is so, though sometimes ignored, is by no means obvious (3.20).

[3] It is important to notice that a quasi-linear representation need not be a map which carries each triangle of the subdivision into a nondegenerate triangle. The point set covered by a quasi-linear map may, for example, be a single point.

is contained in the sphere of diameter ϵ about a. Therefore $T^*(S)$ is in the same sphere. Now in view of our choice of α it follows that $\|x^*(u)-x(u)\|<\epsilon$ for $u\in Q$.

REMARK. The theorem may be stated with any rectangle R replacing the square Q.

Let us turn at once to see how this stretching process is applied in the proof of the additivity result.

Consider a continuous triple T: $x(u)$, $u\in Q$. Let Q be divided by a line $u^1=c$ into two rectangles R_1 $[0\leqq u^1\leqq c,\ 0\leqq u^2\leqq 1]$ and R_2 $[c\leqq u^1\leqq 1,\ 0\leqq u^2\leqq 1]$. Let T_i be the triple $x(u)$, $u\in R_i$, $i=1, 2$. Now T is a representation of a surface $\mathfrak{S}$, while T_i is a representation of a surface $\mathfrak{S}_i$, $i=1, 2$. It is both well known and easy to see that $L(\mathfrak{S})\geqq L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$. The reverse inequality is usually false. We are interested in a case in which it is true, making the Lebesgue area additive.

THEOREM 2. *If $x(u)$ is constant on $u^1=c$, then $L(\mathfrak{S})=L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$.*

PROOF. Grant, for the moment, the existence of two sequences of polyhedra $\{{}_1\mathfrak{P}_n\}$ and $\{{}_2\mathfrak{P}_n\}$, having the property that for $i=1, 2$, $E({}_i\mathfrak{P}_n)\to L(\mathfrak{S}_i)$, and each ${}_i\mathfrak{P}_n$ has a quasi-linear representation ${}_ix_n(u)$ on R_i such that $\|{}_ix_n(u)-x(u)\|<1/n$ for $u\in R_i$.

The justification of this will be taken up presently.

It is enough to obtain a sequence of polyhedra $\{\mathfrak{P}_n\}$ with quasi-linear representation $x_n(u)$ on Q such that $E(\mathfrak{P}_n)$ is within $1/n$ of $E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)$, and $\mathfrak{P}_n\to\mathfrak{S}$.

As a matter of fact, we are going to obtain the stronger result, $E(\mathfrak{P}_n)=E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)$, and $\mathfrak{P}_n\to\mathfrak{S}$.

The theorem will then follow, since by the definition of the Lebesgue area, $L(\mathfrak{S})\leqq\liminf E(\mathfrak{P}_n)=\liminf [E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)]=\lim E({}_1\mathfrak{P}_n)+\lim E({}_2\mathfrak{P}_n)=L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$.

The process by which the sequence of polyhedra $\{\mathfrak{P}_n\}$ is obtained is based on Theorem 1. Were we to define $x_n(u)$ to be ${}_ix_n(u)$ on R_i the $x_n(u)$ would fail to represent the required sequence of polyhedra only by virtue of the fact that continuity, in general, would be lacking along $u^1=c$. Theorem 1 essentially enables us to alter ${}_1x_n(u)$ and ${}_2x_n(u)$ *slightly as to position* and *not at all as to area* so that they will match along $u^1=c$.

Consider any ${}_i\mathfrak{P}_n$ and its quasi-linear representation ${}_ix_n(u)$, $u\in R_i$. Theorem 1 guarantees the existence of a polyhedron ${}_i\mathfrak{P}_n^*$ with quasi-linear representation ${}_ix_n^*(u)$, $u\in R_i$, such that:

(1) $E({}_i\mathfrak{P}_n^*)=E({}_i\mathfrak{P}_n)$.

(2) ${}_ix_n^*(u)=x(u)$, a constant, on $[u^1=c,\ 0\leqq u^2\leqq 1]$.

(3) $\|{}_ix_n^*(u) - {}_ix_n(u)\| < 2/n$.

Now define $T_n: x_n(u) = {}_ix_n^*(u)$, $u \in R_i$, $i = 1, 2$.

This is a *continuous* triple due to condition (2), and so it is a quasi-linear triple representing a polyhedron $\mathfrak{P}_n$. We shall show that $\{\mathfrak{P}_n\}$ has all the desired properties.

First, $E(\mathfrak{P}_n) = E({}_1\mathfrak{P}_n) + E({}_2\mathfrak{P}_n)$ since the elementary area is additive.

Next, for $i = 1, 2$:

$$\begin{aligned}\|x_n(u) - x(u)\| &= \|{}_ix_n^*(u) - x(u)\| \text{ on } R_i \\ &\leqq \|{}_ix_n^*(u) - {}_ix_n(u)\| + \|{}_ix_n(u) - x(u)\| \text{ on } R_i \\ &\leqq 2/n + 1/n \text{ on } Q.\end{aligned}$$

Therefore $x_n(u)$ converges uniformly to $x(u)$ on Q and hence $\{\mathfrak{P}_n\}$ certainly converges to $\mathfrak{S}$.

It now remains but to consider the statement deferred in the first paragraph of the proof.

Briefly the problem amounts to this: *Given a representation $x(u)$ of a surface $\mathfrak{S}$ on a rectangle R we need to know that there exists a polyhedron $\mathfrak{P}$ whose area is close to the area of $\mathfrak{S}$ and admits of a quasi-linear representation on R which is close to the given representation of $\mathfrak{S}$.*

Given $\epsilon > 0$, by the definition of the Lebesgue area (3.13) there is a polygon $\mathfrak{P}$ such that $d(\mathfrak{P}, \mathfrak{S}) < \epsilon$ and $|E(\mathfrak{P}) - L(\mathfrak{S})| < \epsilon$.

Since $\mathfrak{P}$ is a polyhedron it has a quasi-linear representation $\bar{x}(u)$, $u \in B$. The set B is the closure of a Jordan region with polygonal boundary. The set B may be R but this is by no means certain, or even useful. By the definition of distance (1.17) there is a topological transformation $\tau(R) = B$ such that $\|x(u) - \bar{x}(\tau(u))\| < \epsilon$.

The continuity of $\bar{x}(u)$ guarantees that there is a $\delta > 0$ such that all maps $\mu(R) = B$ satisfying $\|\mu(u) - \tau(u)\| < \delta$ have the property that $\|\bar{x}(\mu(u)) - \bar{x}(\tau(u))\| < \epsilon$. Hence $\|x(u) - \bar{x}(\mu(u))\| < 2\epsilon$.

We can select $\mu(u)$ so that in addition to satisfying the above inequality it is quasi-linear and topological on R. The proof of this assertion follows as a result of some remarks of Franklin and Wiener [1, pp. 764–766].

LEMMA (FRANKLIN AND WIENER). *Given $\epsilon > 0$ and a topological transformation $\tau(s) = S$ (where s and S are closed squares) we may subdivide s and S into corresponding convex polygons · · · having the property that any topological transformation $\mu(s) = S$ which maps each polyhedron of s onto the corresponding polyhedron of S is at a distance less than ϵ from τ.*

The reader will readily make the necessary adjustments from s to R and S to B. The convex polygons mentioned above may be chosen as triangles and a μ may be exhibited which is linear on the triangles of R.

Now $x(\mu(u))$ is clearly quasi-linear on R. Moreover, it is a representation of $\mathfrak{P}$, and so $\mathfrak{P}$ has a quasi-linear representation on R within 2ϵ of the given representation of $\mathfrak{S}$. Since $|E(\mathfrak{P})-L(\mathfrak{S})|<\epsilon$ the proof is complete.

We have incidentally proved the independently important theorem:

THEOREM 3. *Any polyhedron has a quasi-linear representation on a rectangle.*

REMARK. The square Q in the additivity theorem can be replaced by a simply connected Jordan region and its boundary, the dividing line can then be replaced by any crosscut on which the triple is constant.

It is possibly of some interest, in conclusion, to compare the surface stretching done here in Theorem 1 with that of the paper by Radó and Reichelderfer. Briefly, their process stretches a surface *directly towards a fixed point.* Each point of the surface sufficiently close to the fixed point is moved along the ray to the fixed point. The stretching is managed, however, so that the increase in area, though generally positive, is ingeniously kept within bounds. Here the stretching is by no means directly towards the fixed point, and it is precisely the rather circuitous route taken to the point which enables us to keep the added area equal to zero. If a polyhedron undergoes their stretching process it will, in general, no longer be a polyhedron; the stretching done here will preserve the polyhedral character of the surface. On the other hand their stretching process applies to situations which this cannot touch. For example, it will deal with the case in which r_1 is a square interior to Q, the set $r_2=\overline{Q-r_1}$, and the triple is constant on $r_1 \cdot r_2$.

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SOLUTION OF THE "PROBLÈME DES MÉNAGES"

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The *problème des ménages* asks for the number of ways of seating n husbands and n wives at a circular table, men alternating with women, so that no husband sits next to his wife. Despite the considerable literature devoted to this problem (cf. the appended bibliography), the following simple solution seems to have been missed.

It is convenient first to solve two preliminary problems, perhaps of some interest in themselves.

LEMMA 1. *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a row is* ${}_{n-k+1}C_k$.

Let $f(n, k)$ be the desired number. We split the selections into two subsets: those which include the last of the n objects and those which do not. The former are $f(n-2, k-1)$ in number (since further selection of the second last object is forbidden); the latter are $f(n-1, k)$ in number. Hence

$$f(n, k) = f(n-1, k) + f(n-2, k-1),$$

and, combining this with $f(n, 1) = n$, we readily prove by induction that $f(n, k) = {}_{n-k+1}C_k$.

LEMMA 2. *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a circle is* ${}_{n-k}C_k n/(n-k)$.

This differs from the preceding problem only in the imposition of the further restriction that no selection is to include both the first and last objects; and the number of such selections which are otherwise acceptable is $f(n-4, k-2)$. Hence the desired result is $f(n, k) - f(n-4, k-2) = {}_{n-k}C_k n/(n-k)$.

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We now restate the *problème des ménages* in the usual fashion by observing that the answer is $2n!u_n$, where u_n is the number of permutations of $1, \cdots, n$ which do not satisfy any of the following $2n$ conditions: 1 is 1st or 2nd, 2 is 2nd or 3rd, $\cdots$, n is nth or 1st. Now let us select a subset of k conditions from the above $2n$ and inquire how many permutations of $1, \cdots, n$ there are which satisfy all k; the answer is $(n-k)!$ or 0 according as the k conditions are compatible or not. If we further denote by v_k the number of ways of selecting k compatible conditions from the $2n$, we have, by the familiar argument of inclusion and exclusion, $u_n=\sum(-1)^k v_k(n-k)!$. It remains to evaluate v_k, for which purpose we note that the $2n$ conditions, when arrayed in a circle, have the property that only consecutive ones are not compatible. It follows from Lemma 2 that $v_k={}_{2n-k}C_k 2n/(2n-k)$, and hence

$$u_n = n! - \frac{2n}{2n-1}{}_{2n-1}C_1(n-1)! + \frac{2n}{2n-2}{}_{2n-2}C_2(n-2)! - \cdots .$$

From this result it follows without difficulty that $u_n/n! \to e^{-2}$ as $n \to \infty$.

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THEORY OF INTERSECTION OF TWO PLANE CURVES

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1. Introduction. This paper is concerned with the study of the projective differential geometry of two plane curves intersecting at an ordinary point. The method used is similar to the one given by the author[1] in a previous paper.

In §2 we show the existence of two projective invariants I, J determined respectively by the fourth order of the two curves at the point of intersection.

In §3 by introducing a certain covariant triangle of reference associated to the two curves at the point of intersection, we reduce the invariants I, J to simpler form and give them geometrical characterizations.

Finally, in §4 a covariant point is suitably chosen for the unit point of the coordinate system in order to obtain the canonical power series expansions of the two curves at the point of intersection. With reference to the vanishing or nonvanishing of the two invariants I, J we have four different types of expansions. The absolute invariants in the expansions of each type are interpreted geometrically in terms of certain double ratios.

2. Derivation of invariants. Let us establish a projective coordinate system in a plane, in which a point has nonhomogeneous coordinates x, y and homogeneous coordinates x_1, x_2, x_3, connected by the relations $x=x_2/x_1$, $y=x_3/x_1$. The context will show in any instance which coordinates are being used. Then let us consider two curves C, $\bar{C}$ in the plane which intersect at an ordinary point O with distinct tangents t, $\bar{t}$. If we choose the point O to be the vertex (1, 0, 0) of the triangle of reference, and the tangents t, $\bar{t}$ to be the sides $y=0$, $x=0$ of the triangle, then the power series expansions of the curves C, $\bar{C}$ in the neighborhood of the point O may be written in the form

$$(1)\qquad \begin{cases} C: & y = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \cdots, \\ \bar{C}: & x = \bar{a}_1y^2 + \bar{a}_2y^3 + \bar{a}_3y^4 + \bar{a}_4y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0.$$

Let us now make a most general transformation of coordinates which shall leave the point O and the tangents t, $\bar{t}$ unchanged. This

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[1] C. C. Hsiung, *Projective differential geometry of a pair of plane curves*, to appear in the Duke Mathematical Journal.

transformation is expressed in terms of the nonhomogeneous coordinates by the equations

$$(2)\qquad \begin{cases} x = B_2x^*/(1 + A_2x^* + A_3y^*), \\ y = C_3y^*/(1 + A_2x^* + A_3y^*). \end{cases}$$

The effect of this transformation on equations (1) is to produce two other equations of the same form each of whose first three coefficients, indicated by stars, are given by the formulas

$$(3)\qquad \begin{cases} C_3a_1^* = B_2^2a_1, \\ C_3(A_2a_1^* - a_2^*) = B_2^2(2A_2a_1 - B_2a_2), \\ C_3(A_2^2a_1^* - A_3a_1^{*2} - A_2a_2^* + a_3^*) = B_2^2(3A_2^2a_1 - 2A_3a_1a_1^* \\ \qquad\qquad - 3A_2B_2a_2 + B_2^2a_3); \\ B_2\bar{a}_1^* = C_3^2\bar{a}_1, \\ B_2(A_3\bar{a}_1^* - \bar{a}_2^*) = C_3^2(2A_3\bar{a}_1 - C_3\bar{a}_2), \\ B_2(A_3^2\bar{a}_1^* - A_2\bar{a}_1^{*2} - A_3\bar{a}_2^* + \bar{a}_3^*) = C_3^2(3A_3^2\bar{a}_1 - 2A_2\bar{a}_1\bar{a}_1^* \\ \qquad\qquad - 3A_3C_3\bar{a}_2 + C_3^2\bar{a}_3). \end{cases}$$

Solving the first four of equations (3) for A_2, A_3, B_2, C_3 and subsituting the results in the fifth, it is easily seen that

$$(4)\qquad I = (1/a_1(a_1\bar{a}_1^2)^{1/3})(a_1\bar{a}_2/\bar{a}_1 + a_2^2/a_1^2 - a_3/a_1)$$

is an invariant. In a way similar to the foregoing we have from the last one of equations (3) another invariant

$$(5)\qquad J = (1/\bar{a}_1(a_1^2\bar{a}_1)^{1/3})(\bar{a}_1a_2/a_1 + \bar{a}_2^2/\bar{a}_1^2 - \bar{a}_3/\bar{a}_1),$$

which can be obtained also immediately by interchanging a_i, $\bar{a}_i$ $(i=1, 2, 3)$ from equation (4).

3. **The canonical triangle and geometrical characterizations of the invariants** I, J. By suitable choice of the other two vertices of the triangle of reference, expansions (1) can be simplified. For this purpose we consider the pencil of conics having contact of the third order with the curve C at the point O. The equation of a general conic of this pencil is readily found to be

$$(6)\qquad a_1^2x^2 + a_2xy - a_1y + ky^2 = 0,$$

where k is arbitrary. The coordinates of the pole O_1 of the tangent $\bar{l}$, $x=0$, with respect to this conic are

$$(a_2, a_1, 0). \tag{7}$$

Similarly, the pole O_2 of the tangent t, $y=0$, with respect to any conic having contact of the third order with the curve $\overline{C}$ at the point O has the coordinates

$$(\bar{a}_2, 0, \bar{a}_1). \tag{8}$$

It is convenient to call the points O_1, O_2 *the associate points* of the curves C, $\overline{C}$ at the point O. If these points be taken for the vertices $(0, 1, 0)$, $(0, 0, 1)$ of the triangle of reference, then $a_2=\bar{a}_2=0$ and therefore equations (1) become

$$\begin{cases} C: & y = a_1x^2 + a_3x^4 + a_4x^5 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + \bar{a}_3y^4 + \bar{a}_4y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{9}$$

while the invariants I, J take the simpler form

$$I = -a_3/a_1^2(a_1\bar{a}_1)^{1/3}, \tag{10}$$

$$J = -\bar{a}_3/\bar{a}_1^2(a_1\bar{a}_1)^{1/3}. \tag{11}$$

Thus the coordinate system except the unit point has been determined, and the triangle OO_1O_2 will be called *the canonical triangle* of the curves C, $\overline{C}$ at the point O.

We are now in position to characterize geometrically the invariants I, J by expressing them in terms of certain double ratios. To this end let us consider the conic which passes through the associate point O_2 and has contact of the third order with the curve C at the point O. Hereafter we shall call this conic *the associate conic* of the curve C at the point O. From equation (6) its equation with respect to the canonical triangle

$$a_1x^2 - y = 0 \tag{12}$$

follows immediately. Similarly, we obtain the equation of the associate conic of the curve $\overline{C}$ at the point O,

$$\bar{a}_1y^2 - x = 0. \tag{13}$$

The two associate conics (12), (13) intersect, besides in O, in three points P_i $(i=1, 2, 3)$ with coordinates

$$(\epsilon^i/(a_1^2\bar{a}_1)^{1/3}, \epsilon^{2i}/(a_1\bar{a}_1^2)^{1/3}), \tag{14}$$

where $\epsilon^3=1$. Let P_k' be the intersections of the lines OP_k, P_iP_j, where $i, j, k=1, 2, 3$ and are all distinct. It is evident that $P_1'P_2'$ is the

unique real one among the three lines $P_i' P_j'$. Its equation may be easily found to be

$$1 - 2(a_1^2\bar{a}_1)^{1/3}x - 2(a_1\bar{a}_1^2)^{1/3}y = 0, \tag{15}$$

and we shall call it *the real associate line* of the curves C, $\overline{C}$ at the point O. This real associate line intersects the tangent $\bar{t}$, $x=0$, in the point R,

$$(0,\ 1/2(a_1\bar{a}_1^2)^{1/3}). \tag{16}$$

On the other hand we consider a general cubic curve which has contact of the fourth order with the curve C at the point O and a node at O with t, $\bar{t}$ for nodal tangents. A simple calculation suffices to give the equation of this nodal cubic curve, namely,

$$a_1^3x^3 + a_3xy^2 - a_1^2xy + ky^3 = 0, \tag{17}$$

where k is arbitrary. The line containing the three inflexions of the cubic (17) has the equation

$$y = a_1^2/a_3, \tag{18}$$

which is intersected by the tangent $\bar{t}$, $x=0$, in the point S,

$$(0,\ a_1^2/a_3). \tag{19}$$

From equations (10), (16), (19) it follows at once that the double ratio of the four points O, O_2, R, S is equal to

$$(OO_2,\ RS) = -I/2. \tag{20}$$

Hence we obtain the following geometrical characterization of the invariant I.

Let two plane curves C, $\overline{C}$ intersect at an ordinary point O with distinct tangents t, $\bar{t}$. Let O_2 be the associate point of the curve $\overline{C}$ at the point O, and C_3 a general cubic curve which has contact of the fifth order with the curve C at the point O and a node at O with t, $\bar{t}$ for nodal tangents. If the tangent $\bar{t}$ intersects the real associate line of the curves C, $\overline{C}$ at O in point R, and intersects the line containing the three inflexions of the nodal cubic curve C_3 in point S; then the invariant I associated with the point O of the curves C, $\overline{C}$ is, except for sign, equal to two times the double ratio $(OO_2,\ RS)$.

The other invariant J can be characterized geometrically in a similar manner.

4. **Canonical power series expansions.** In this section we shall com-

plete the characterization of the coordinate system by suitably choosing a covariant point for the unit point, and obtain the canonical power series expansions of the curves C, $\overline{C}$ in the neighborhood of the point O. It will be convenient to discuss four cases with reference to the vanishing or nonvanishing of the two invariants I, J.

Case I. $IJ \neq 0$. Here we can choose for the unit point the intersection of the line of inflexions (18) and the similar one

$$x = \bar{a}_1^2/\bar{a}_3, \tag{21}$$

so that $a_3 = a_1^2$, $\bar{a}_3 = \bar{a}_1^2$. Thus we arrive at *the canonical power series expansions of the curves C, $\overline{C}$ in the neighborhood of the point O,*

$$\begin{cases} C: & y = a_1x^2 + a_1^2x^4 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + \bar{a}_1^2y^4 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{22}$$

in which the coefficients a_1, $\bar{a}_1$ can be expressed in terms of the two invariants I, J, namely,

$$a_1 = -I/J^2, \qquad \bar{a}_1 = -J/I^2. \tag{23}$$

Case II. $I = 0$, $J \neq 0$. In this case we observe that in the pencil (17) of nodal cubic curves there is a unique one having contact of the fifth order with the curve C at the point O. Making use of the first of equations (9) and noticing that $a_3 = 0$ we find at once the equation of this cubic, namely,

$$a_1^4x^3 + a_4y^3 - a_1^3xy = 0. \tag{24}$$

The join m of the two real intersections of the cubic curve (24) and the associate conic (13) of the curve $\overline{C}$ at O is given by the equation

$$a_1a_1^{1/3}x + (a_4 - a_1^3\bar{a}_1)^{1/3}y = 0. \tag{25}$$

If the unit point of the coordinate system be taken at the intersection of the lines (21), (25), then $a_4 = a_1^2(\bar{a}_1 - a_1)$, $\bar{a}_3 = \bar{a}_1^2$, and therefore *the canonical power series expansions of the curves C, $\overline{C}$ in the neighborhood of the point O* are reached:

$$\begin{cases} C: & y = a_1x^2 + a_1^3(\bar{a}_1 - a_1)x^5 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + \bar{a}_1^2y^4 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0. \tag{26}$$

Accordingly, the invariant J becomes

$$J = -1/(a_1^2\bar{a}_1)^{1/3}. \tag{27}$$

In §2 we have shown that there are only two independent invari-

ants determined by the neighborhoods up to and including the fourth order of the curves C, $\bar{C}$ at the point O, one of them vanishing now. In order to exhaust the two independent coefficients in the canonical expansions (26) it is sufficient for us to find another independent invariant of the curves C, $\bar{C}$ at the point O. For this purpose let r be the join of the two real intersections of the two associate conics (12), (13), whose equation is found to be

$$a_1^{1/3}x - \bar{a}_1^{1/3}y = 0. \tag{28}$$

From equations (25), (28) the following double ratio is immediate:

$$\Delta \equiv (it, mr) = (\bar{a}_1/a_1)^{1/3}. \tag{29}$$

Thus the two coefficients a_1, $\bar{a}_1$ in expansions (26) have been completely expressed in terms of the invariant J and the double ratio Δ, namely,

$$a_1 = -1/J\Delta, \qquad \bar{a}_1 = -\Delta^2/J. \tag{30}$$

Case III. $I\neq 0$, $J=0$. Similarly, with the rôles of the curves C, $\bar{C}$ in Case II interchanged we obtain *the desired canonical power series expansions for this case,*

$$\begin{cases} C\colon\ y = a_1x^2 + a_1^2x^4 + \cdots, \\ \bar{C}\colon\ x = \bar{a}_1y^2 + \bar{a}_1^3(a_1 - \bar{a}_1)y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{31}$$

in which the two coefficients a_1, $\bar{a}_1$ can be expressed also in terms of the invariant I and a double ratio similar to (29).

Case IV. $I=0$, $J=0$. It is useful for this case to consider the cubic curve which has contact of the fifth order with the curve $\bar{C}$ at the point O and a node at O with t, $\bar{t}$ for nodal tangents. The equation of this cubic can be written immediately by interchanging x, y and a_i, $\bar{a}_i$ $(i=1, 4)$ from equation (24). The result is

$$\bar{a}_4x^3 + \bar{a}_1^4y^3 - \bar{a}_1^3xy = 0. \tag{32}$$

The line n joining the associate point O_2 to the real intersection, other than O, of the cubic curve (32) and the associate conic (12) of the curve C at O is given by the equation

$$x = (1/a_1\bar{a}_1)(a_1\bar{a}_1^2 - \bar{a}_4/\bar{a}_1)^{1/3}. \tag{33}$$

If the unit point of the coordinate system be taken at the intersection of the lines (25), (33), then $a_4=a_1^3$ $(\bar{a}_1-a_1)$, $\bar{a}_4=a_1\bar{a}_1^3(1-a_1^2\bar{a}_1)$, and therefore we reach *the required canonical power series expansions,*

$$(34)\qquad \begin{cases} C: & y = a_1x^2 + a_1^3(\bar{a}_1 - a_1)x^5 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + a_1\bar{a}_1^3(1 - a_1^2\bar{a}_1)y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0.$$

In order to interpret geometrically the coefficients in the canonical expansions (34), let us assume s to be the line joining the associate point O_2 to the real intersection, other than O, of the two associate conics (12), (13), whose equation is found to be

$$(35)\qquad x_1 - (a_1^2\bar{a}_1)^{1/3}x_2 = 0.$$

From equations (33), (35) we obtain at once the following double ratio,

$$(36)\qquad D \equiv (\bar{l}, O_1O_2, n, s) = (a_1^2\bar{a}_1)^{1/3}.$$

Thus the two coefficients a_1, $\bar{a}_1$ in expansions (34) have been completely expressed in terms of the two double ratios Δ, D, namely,

$$(37)\qquad a_1 = D/\Delta, \qquad \bar{a}_1 = D\Delta^2.$$

Some of the results of this section are contained in the following summary.

Let two plane curves C, $\overline{C}$ intersect at an ordinary point O with distinct tangents t, $\bar{t}$. The projective differential geometry of the curves C, $\overline{C}$ in the neighborhood of the point O can be studied by means of expansions (22), (26), (31), (34) *of four different types, all of whose coefficients are absolute invariants of the curves. The covariant coordinate system for each type of expansions has the following geometric description. The vertex* (1, 0, 0) *of the triangle of reference is the point O under consideration, and the other vertices* (0, 1, 0), (0, 0, 1) *are the associate points O_1, O_2 of the curves C, $\overline{C}$ at the point O. Let $C_3(\overline{C}_3)$ be a general one of a pencil of cubic curves which have contact of the fourth order with the curve $C(\overline{C})$ at the point O and a node at O with t, $\bar{t}$ for nodal tangents, $K_3(\overline{K}_3)$ the unique cubic of this pencil having contact of the fifth order with the curve $C(\overline{C})$ at the point O, and $l(\bar{l})$ the line of inflexions of the nodal cubic curve $C_3(\overline{C}_3)$. Further, let $m(\bar{m})$ be the join of the two real intersections of the cubic $K_3(\overline{K}_3)$ with the associate conic $\overline{C}_2(C_2)$ of the curve $\overline{C}(C)$ at O, and n the line joining the associate point O_2 to the real intersection, other than O, of the cubic $\overline{K}_3$ with the associate conic C_2. Then the unit point of the covariant coordinate system for each type of expansions is respectively one of the intersections of the four pairs of lines*: (i) *l and $\bar{l}$,* (ii) *$\bar{l}$ and m,* (iii) *l and $\bar{m}$, and* (iv) *m and n.*

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A TRANSFORMATION OF JONAS SURFACES

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It is well known that when an analytic surface S is referred to its asymptotic net (u, v) the homogeneous point coordinates $x^i(u, v)$ $(i=1, 2, 3, 4)$ of a generic point on S can then be normalized, so that they satisfy the differential equations,

$$(1) \qquad \begin{cases} x_{uu} = \beta x_v + px, \\ x_{vv} = \gamma x_u + qx, \end{cases}$$

where the coefficients β, γ, p, q satisfy the conditions of integrability,

$$(2) \qquad \begin{cases} (\beta_v + 2p)_v = (\beta\gamma)_u + \beta\gamma_u, \quad (\gamma_u + 2q)_u = (\beta\gamma)_v + \gamma\beta_v, \\ (p_v + \beta q)_v + \beta_v q = (q_u + \gamma p)_u + \gamma_u p. \end{cases}$$

The conjugate net Ω of S defined by

$$Cdu^2 + Ddv^2 = 0,$$

has equal point invariants when and only when[1]

$$(3) \qquad (\log (C/D))_{uv} - (\gamma(C/D))_v + (\beta(D/C))_u = 0.$$

The necessary and sufficient condition that Ω should have equal tangential invariants is obtained from (3) by replacing β, γ by $-\beta$, $-\gamma$ respectively. If Ω has equal invariants, both point and tangential, then it is a Jonas net, and S then becomes a Jonas surface.[2] For a Jonas net we have thus the following relations:

$$(\log (C/D))_{uv} = 0, \qquad (\gamma(C/D))_u - (\beta(D/C))_v = 0.$$

By a suitable transformation of asymptotic parameters, leaving the asymptotic net unaltered, the above equations reduce to

$$\beta_u = \gamma_v, \qquad C = D.$$

Hence a Jonas net on a Jonas surface S may be represented by the equation

$$(4) \qquad du^2 - dv^2 = 0,$$

and the surface is characterized by

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[1] Cf. G. Fubini-E. Čech, *Geometria Proiettiva Differenziale*, vol. 1, Bologna, Zanichelli, 1927, p. 105.

[2] Cf. Fubini-Čech, ibid. p. 106.

$$\beta_u = \gamma_v. \tag{5}$$

The main object of this note is to prove the following theorem:

THEOREM. *The projection in a fixed plane of a Jonas net of a Jonas surface is a plane net with equal point invariants, and stands for the projection of the asymptotic net of another Jonas surface.*

A point P_θ with the coordinates $\theta = rx + sx_u + tx_v + lx_{uv}$ is fixed in space if

$$r = l_{uv} - l\beta\gamma, \qquad s = -l_v, \qquad t = -l_u, \tag{6}$$

where l satisfies the system of equations

$$\begin{cases} l_{uu} = -\beta l_v + (\beta_v + p)l, \\ l_{vv} = -\gamma l_u + (\gamma_u + q)l. \end{cases} \tag{7}$$

A point P_y on the straight line $P_\theta P_x$, is evidently given by $y = \lambda\theta + x$. In virtue of (1), we find by differentiation that

$$\begin{cases} y = (1+\lambda r)x + \lambda s x_u + \lambda t x_v + \lambda l x_{uv}, \\ y_u = \lambda_u r x + (1+\lambda_u s)x_u + \lambda_u t x_v + \lambda_u l x_{uv}, \\ y_v = \lambda_v r x + \lambda_v s x_u + (1+\lambda_v t)x_v + \lambda_v l x_{uv}, \\ y_{uv} = \lambda_{uv} r x + \lambda_{uv} s x_u + \lambda_{uv} t x_v + (1+\lambda_{uv} l)x_{uv}, \\ y_{uu} = (\lambda_{uu} r + p)x + \lambda_{uu} s x_u + (\lambda_{uu} t + \beta)x_v + \lambda_{uu} l x_{uv}, \\ y_{vv} = (\lambda_{vv} r + q)x + (\lambda_{vv} s + \gamma)x_u + \lambda_{vv} t x_v + \lambda_{vv} l x_{uv}. \end{cases} \tag{8}$$

In order that P_y be in a fixed plane, it is necessary and sufficient that

$$\begin{cases} y_{uu} = A y_u + B y_v + C y, \\ y_{uv} = A' y_u + B' y_v + C' y, \\ y_{vv} = A'' y_u + B'' y_v + C'' y. \end{cases} \tag{9}$$

Substituting (8) in (9) and reducing, we obtain

$$\begin{aligned} A &= 0, & B &= \beta, & C &= p, \\ A' &= (\log l)_v, & B' &= (\log l)_u, & C' &= \beta\gamma - l_{uv}/l, \\ A'' &= \gamma, & B'' &= 0, & C'' &= q, \end{aligned}$$

and the conditions for the parameter λ

$$\begin{aligned} \lambda_{uu} &= \beta\lambda_v + p\lambda, \\ \lambda_{vv} &= \gamma\lambda_u + q\lambda, \\ r\lambda + s\lambda_u + t\lambda_v + l\lambda_{uv} + 1 &= 0. \end{aligned}$$

Thus we have a plane net given by the equations

$$(10)\qquad \begin{cases} y_{uu} = \beta y_v + py, \\ y_{vv} = \gamma y_u + qy, \\ y_{uv} = (\log l)_v y_u + (\log l)_u y_v + (\beta\gamma - l_{uv}/l)y. \end{cases}$$

The curves of this net are the perspectives on the fixed plane of the asymptotic curves of a Jonas surface S obtained by projecting from the centre P_θ. In order to obtain the perspectives on the same fixed plane of the Jonas net Ω, we have to use the transformation

$$\bar{u} = u - v, \qquad \bar{v} = u + v,$$

so that

$$(11)\qquad \begin{cases} y_{\bar{u}} = (y_u - y_v)/2, \\ y_{\bar{v}} = (y_u + y_v)/2, \end{cases}$$

namely,

$$(11')\qquad \begin{cases} y_u = y_{\bar{u}} + y_{\bar{v}}, \\ y_v = y_{\bar{v}} - y_{\bar{u}}. \end{cases}$$

Putting

$$\log l = \theta, \qquad \beta\gamma - l_{uv}/l = c,$$

we find after a simple calculation that

$$(12)\qquad \begin{cases} y_{\bar{u}\bar{u}} = (1/4)(\gamma - \beta + 2\theta_u - 2\theta_v)y_{\bar{u}} \\ \qquad\quad + (1/4)(\gamma + \beta - 2\theta_v - 2\theta_u)y_{\bar{v}} + (1/4)(p + q - 2c)y, \\ y_{\bar{v}\bar{v}} = (1/4)(\gamma - \beta - 2\theta_u + 2\theta_v)y_{\bar{u}} \\ \qquad\quad + (1/4)(\gamma + \beta + 2\theta_v + 2\theta_u)y_{\bar{v}} + (1/4)(p + q + 2c)y, \\ y_{\bar{u}\bar{v}} = -(1/4)(\beta + \gamma)y_{\bar{u}} + (1/4)(\beta - \gamma)y_{\bar{v}} + (1/4)(p - q)y, \end{cases}$$

which represent the perspectives of the Jonas net $du^2 - dv^2 = 0$. Since

$$-(\beta + \gamma)_{\bar{u}} = (\beta - \gamma)_{\bar{v}},$$

this net is of equal point invariants and therefore asymptotic. That is, it may be regarded as the perspectives of the asymptotic curves of a certain surface Q. The projective linear element of the surface Q is easily found to be

$$(\bar{\beta}d\bar{u}^3 + \bar{\gamma}d\bar{v}^3)/2d\bar{u}d\bar{v},$$

where[3]

[3] The projective linear element of a plane net has been defined by E. Čech. Cf. Fubini-Čech, *Introduction à la géométrie projective différentielle des surfaces*, 1931, chap. 10.

$$\bar{\beta} = (\gamma + \beta - 2\theta_v - 2\theta_u)/4, \qquad \bar{\gamma} = (\gamma - \beta - 2\theta_u + 2\theta_v)/4.$$

The relation $\beta_u = \gamma_v$ gives, however, the similar relation

$$\bar{\beta}_{\bar{u}} = \bar{\gamma}_{\bar{v}}.$$

Hence Q is also a Jonas surface, which completes the proof.

As a special case of the theorem we have the result:

Any Jonas net of a Jonas surface S is perspective to the asymptotic net of another Jonas surface Q from a fixed point. Conversely, if a Jonas net of a Jonas surface is perspective to the asymptotic net of another surface Q, then Q is also a Jonas surface.

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